

## Gravity duals of half-BPS Wilson loops

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**ABSTRACT:** We explicitly construct the fully back-reacted half-BPS solutions in Type IIB supergravity which are dual to Wilson loops with 16 supersymmetries in  $\mathcal{N} = 4$  super Yang-Mills. In a first part, we use the methods of a companion paper to derive the exact general solution of the half-BPS equations on the space  $AdS_2 \times S^2 \times S^4 \times \Sigma$ , with isometry group  $SO(2,1) \times SO(3) \times SO(5)$  in terms of two locally harmonic functions on a Riemann surface  $\Sigma$  with boundary. These solutions, generally, have varying dilaton and axion, and non-vanishing 3-form fluxes. In a second part, we impose regularity and topology conditions. These non-singular solutions may be parametrized by a genus  $g \geq 0$  hyperelliptic surface  $\Sigma$ , all of whose branch points lie on the real line. Each genus  $g$  solution has only a single asymptotic  $AdS_5 \times S^5$  region, but exhibits  $g$  homology 3-spheres, and an extra  $g$  homology 5-spheres, carrying respectively RR 3-form and RR 5-form charges. For genus 0, we recover  $AdS_5 \times S^5$  with 3 free parameters, while for genus  $g \geq 1$ , the solution has  $2g + 5$  free parameters. The genus 1 case is studied in detail. Numerical analysis is used to show that the solutions are regular throughout the  $g = 1$  parameter space. Collapse of a branch cut on  $\Sigma$  subtending either a homology 3-sphere or a homology 5-sphere is non-singular and yields the genus  $g - 1$  solution. This behavior is precisely expected of a proper dual to a Wilson loop in gauge theory.

**KEYWORDS:** AdS-CFT Correspondence, Brane Dynamics in Gauge Theories.

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## 1. Introduction

In the AdS/CFT correspondence, [1–3] (for reviews, see [4, 5]) a prominent role is played by objects which are protected by supersymmetry. Non-renormalization theorems make

comparison of weak and strong coupling calculations possible and BPS equations often provide an easier way to obtain exact solutions of the equations of motion.

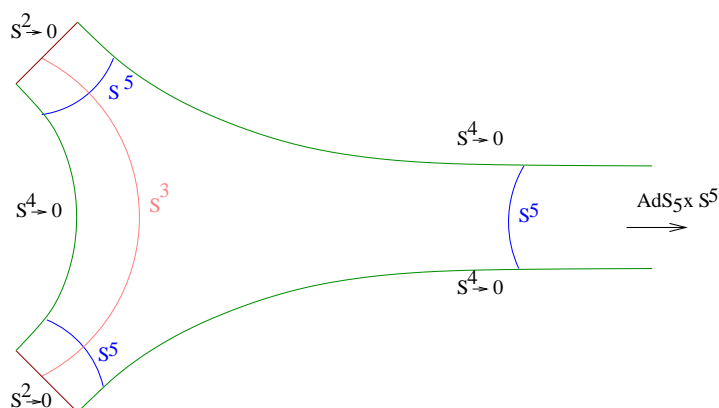
The first example of such objects is provided by local gauge invariant chiral primary operators  $\mathcal{O}_J$  in  $\mathcal{N} = 4$  super Yang-Mills theory (SYM) with gauge group  $SU(N)$ , where  $J$  denotes a particular  $U(1)$  charge of the  $SO(6)$  R-symmetry. Supersymmetry protects the conformal dimension  $\Delta$  of these operators against quantum corrections, so that we have the exact relation  $\Delta = J$ . For  $\Delta \ll N$ , these operators are dual to small fluctuations of supergravity modes in the  $AdS_5 \times S^5$  bulk [3]. For  $\Delta \sim N$ , they can be associated with giant gravitons (probe branes on  $S^5$  or  $AdS_5$ ) [6–8]. For  $\Delta \sim N^2$ , the fully back-reacted geometries preserving  $SO(4) \times SO(4)$  as well as 16 of the 32 supersymmetry were found in [9] and referred to as “bubbling AdS”. All regular solutions are parameterized by the “coloring” of  $\mathbf{R}^2$  in black and white regions.

In two recent papers [10, 11], the present authors constructed general half-BPS solutions with  $SO(2, 3) \times SO(3) \times SO(3)$  symmetry in Type IIB supergravity. The solutions are given by a warping of  $AdS_4 \times S^2 \times S^2$  over a two-dimensional surface  $\Sigma$ , and have varying dilaton and axion, as well as non-vanishing NSNS and RR 3-form fluxes. They generalize the non-supersymmetric [12] and  $\mathcal{N} = 1$  supersymmetric [13] Janus solutions. These solutions are holographic duals of (generalized) interface SYM theories [14, 15].

The solutions were found by solving the BPS equations of Type IIB supergravity for the most general  $SO(2, 3) \times SO(3) \times SO(3)$  symmetric Ansatz on the manifold  $AdS_4 \times S^2 \times S^2 \times \Sigma$  (see also [16]). The solutions are parametrized by two harmonic functions  $h_1, h_2$  on a genus  $g$  hyperelliptic Riemann surface  $\Sigma$  with boundary, with all the branch cuts restricted to lie on the real axis. The regularity of the solutions imposes various conditions on the harmonic functions. We refer the reader to the papers [10, 11] for details. The choice of  $g + 1$  branch cuts along the real axis is the one-dimensional analog of the coloring of the “bubbling AdS” solution of [9].

Another important class of operators consists of Wilson loops corresponding to the holonomies of gauge fields along (closed) contours. The AdS dual of a Wilson loop operator in the fundamental representation was identified in [17, 18], with a fundamental string world sheet in the bulk of AdS which ends on the contour of the Wilson loop on the boundary of AdS. In particular, we will be interested in half-BPS Wilson loop operators. Proposals for the AdS-dual description of half-BPS Wilson loop operators in higher dimensional representations of  $SU(N)$  have been made in [19–23].

In this paper, the fully back-reacted supergravity solutions corresponding to half-BPS Wilson loops will be derived and explicit formulas for the solutions will be presented. We shall follow closely the methods developed for the half-BPS Type IIB interface solutions in [10, 11]. In particular, we shall solve the BPS equations for our Ansatz explicitly in terms of two harmonic functions  $h_1, h_2$  defined on a two-dimensional Riemann surface  $\Sigma$ . Indeed, many formulas in the present paper will be very similar or identical to the ones in [10, 11]. There are, however, subtle and important differences between the solutions. Furthermore, while it is possible to formally relate the solutions on  $AdS_4 \times S^2 \times S^2 \times \Sigma$  of [10, 11] to the solutions on  $AdS_2 \times S^2 \times S^4 \times \Sigma$  given in the present paper by an analytic continuation, it is not a priori guaranteed that both sets of solutions will preserve the



**Figure 1:** The genus 1 solution with homology 3-spheres and 5-spheres.

same number of supersymmetries or even that regular solutions will be mapped to regular solutions. A supergravity description of half-BPS Wilson loops has already been given in [24] (See also [22] for an earlier attempt at a solution), using the Killing tensor methods of [9, 25, 26]. The solution found in [24] was parameterized by a harmonic function but some quantities were only implicitly given in terms of the harmonic function.

A summary of the key properties of our AdS solutions is as follows. The  $SO(2, 1) \times SO(3) \times SO(5)$ -invariant metrics on  $AdS_2 \times S^2 \times S^4 \times \Sigma$  are parametrized by

$$ds^2 = f_1^2 ds_{AdS_2}^2 + f_2^2 ds_{S^2}^2 + f_4^2 ds_{S^4}^2 + ds_{\Sigma}^2 \tag{1.1}$$

The metrics  $ds_{AdS_2}^2$ ,  $ds_{S^2}^2$ , and  $ds_{S^4}^2$  correspond to unit radii. The metric  $ds_{\Sigma}^2$  is positive, and may be expressed as  $ds_{\Sigma}^2 = 4\rho^2 |dw|^2$  in terms of local complex coordinates  $w, \bar{w}$  on the Riemann surface  $\Sigma$  with boundary  $\partial\Sigma$ . The dilaton  $\phi$ , and  $\rho, f_1, f_2, f_4$  are real functions on  $\Sigma$ . All half-BPS solutions may be expressed in terms of two real harmonic functions  $h_1$  and  $h_2$  on  $\Sigma$ . The dilaton and metric functions for these solutions are given by the following relations,

$$e^{4\phi} = - \frac{2h_1 h_2 |\partial_w h_2|^2 - h_2^2 (\partial_w h_1 \partial_{\bar{w}} h_2 + \partial_w h_2 \partial_{\bar{w}} h_1)}{2h_1 h_2 |\partial_w h_1|^2 - h_1^2 (\partial_w h_1 \partial_{\bar{w}} h_2 + \partial_w h_2 \partial_{\bar{w}} h_1)} \tag{1.2}$$

as well as

$$\begin{aligned} f_2^2 f_4^2 &= 4h_2^2 e^{-2\phi} \\ f_1^2 f_4^2 &= 4h_1^2 e^{+2\phi} \end{aligned} \tag{1.3}$$

Explicit expressions for the solutions of  $\rho, f_1$  are given in (7.18) and (7.26). (The Ansatz for the antisymmetric tensor fields, will be given in (7.28), (7.29), (7.30), and their solutions obtained in section 7.6.)

For all regular solutions, the functions  $f_1, f_2, f_4$ , and  $e^{\pm 2\phi}$  are non-vanishing inside  $\Sigma$ . The boundary  $\partial\Sigma$  has either the  $S^2$  or the  $S^4$  shrink to 0 radius, so that either  $f_2$  or  $f_4$  vanishes respectively, and  $h_2 = 0$  throughout  $\partial\Sigma$ . Since  $f_1 \neq 0$  on  $\partial\Sigma$ , we also have  $h_1 = 0$

on segments of  $\partial\Sigma$  if and only if the  $S^4$  shrinks to zero on that segment. All solutions have a single asymptotic  $AdS_5 \times S^5$  region. A schematic picture for the case where  $\Sigma$  is a genus 1 surface is given in figure 1. The presence of a non-trivial homology 3-sphere indicates the presence of a non-vanishing RR 3-form charge as will be computed in section 10.5.

The plan of this paper is as follows. In section 2, we review half-BPS Wilson loops in the context of AdS/CFT. In section 3, we present our Ansatz as  $AdS_2 \times S^2 \times S^4$  warped over a two dimensional space  $\Sigma$ . The general ten-dimensional Killing spinor is decomposed with respect to the  $AdS_2 \times S^2 \times S^4$  factors. In section 4, the BPS equations are reduced by utilizing the Killing spinors. In section 5, reality conditions are imposed on the solution and the BPS equations are reduced to equations on a two dimensional complex spinor. In section 6, these equations are reduced to an integrable system. In section 7, the integrable system is mapped to a first order system which can be solved in terms of two harmonic functions. The explicit form of the metric factor, dilaton and three form fluxes in terms of the harmonic functions are given. In section 8, it is shown that the only solution with a constant dilaton is  $AdS_5 \times S^5$ . In section 9, the conditions for obtaining regular solutions are derived and the boundary conditions on the harmonic functions are derived. In section 10, a general class of regular solutions is constructed with a genus  $g$  hyperelliptic surface. In section 11, the genus 1 case is discussed in detail and all the quantities are explicitly expressed in terms of elliptic functions. In section 12, we study the collapse of a branch cut on  $\Sigma$  between consecutive branch points, and show that the genus  $g$  solution collapses to a regular genus  $g - 1$  solution. In three appendices we give our conventions for the Clifford algebra, Killing spinors, Bianchi identities and equations of motion.

## 2. Wilson loops in gauge theory and supergravity

The Wilson loop operator is an important gauge invariant observable in gauge theories. For  $\mathcal{N} = 4$  super Yang-Mills theories, the appropriate operator is defined as [17, 18, 27]

$$W_R(C) = \text{Tr}_R P \exp \left( i \int_C d\tau (A_\mu \dot{x}^\mu + \phi_I \dot{y}^I) \right) \quad (2.1)$$

Here,  $\text{Tr}_R$  labels the trace over an arbitrary representation  $R$  of  $SU(N)$  and  $(x^\mu(\tau), y^I(\tau))$  parameterizes a path in  $R^{1,3} \times R^6$  coupling the curve  $C$  to the gauge field and the six adjoint scalars of  $\mathcal{N} = 4$  SYM. The Wilson loop operator can preserve some supersymmetry. In particular, it was shown in [27] that the preservation of eight Poincaré supersymmetries restricts the path in  $R^{1,9}$  to be null, i.e.  $\dot{x}^2 + \dot{y}^2 = 0$ . The preservation of eight superconformal symmetries furthermore fixes the trajectory on  $R^{1,3}$  to be a timelike line, i.e.  $x^0 = \tau, x^i = 0$  and the trajectory on  $R^6$  to be given by  $\dot{y}^I = n^I$ , where  $n^I$  is a unit vector in  $R^6$ . Hence the half-BPS Wilson loop operator in the representation  $R$  becomes

$$W_R(C) = \text{Tr}_R P \exp \left( i \int_C d\tau (A_0 + n_I \phi^I) \right) \quad (2.2)$$

### 2.1 Symmetries of the half-BPS Wilson loop

In the following, we review the derivation of the supergroup preserved by the half-BPS Wilson loop. The choice of the unit vector  $n^I$  breaks the  $SO(6)$  R-symmetry to  $SO(5)$ .

The superconformal symmetry  $SO(4, 2)$  is broken by the timelike straight line as follows. The condition  $x^i = 0$ ,  $i = 1, 2, 3$  is left invariant by  $SO(3)$  spatial rotations. The condition  $\dot{x}^0 = 1$  is invariant under time translations, dilations and special conformal transformations which together generate  $SO(2, 1)$ . Hence the residual bosonic symmetry is

$$SO(2, 1) \times SO(3) \times SO(5) \tag{2.3}$$

The sixteen unbroken supersymmetries transform under the  $(4, 4)$  of the bosonic symmetry group and form a supergroup  $OSP(4^*|4)$ .

For the supergravity description of the half-BPS Wilson loops, we seek a general Ansatz in Type IIB supergravity with the above symmetry. The factor  $SO(2, 1)$  requires the geometry to contain  $AdS_2$ , the factor  $SO(3)$  requires  $S^2$ , and the factor  $SO(5)$  requires  $S^4$ . Two dimensions remain undetermined by the symmetries alone, so that the most general space of interest to us will be of the form,

$$AdS_2 \times S^2 \times S^4 \times \Sigma \tag{2.4}$$

where  $\Sigma$  stands for the two-dimensional parameter space, over which the above products are warped.

## 2.2 Geometry of fluxes

The holographic dual description of a (probe) Wilson loop operator in the fundamental representation of  $SU(N)$  is given by a string worldsheet in the  $AdS_5 \times S^5$  bulk which ends on the contour  $C$  of the Wilson loop on the boundary of  $AdS_5 \times S^5$ . The holographic description of half-BPS Wilson loop operators in higher dimensional representations of  $SU(N)$  was developed by several authors [19–22]. It appears that there are two equivalent descriptions in terms of D-branes with fundamental string charge.

In the first proposal [20–22], a Wilson loop in the  $k$ -th symmetric tensor representation (which is labeled by a Young-tableau of a row of  $k$  boxes) is given by a D3 brane with  $AdS_2 \times S^2$  worldvolume with  $k$  units of fundamental string charge dissolved on the  $AdS_2$ . A general Young-tableau with  $p$  rows with  $n_i$ ,  $i = 1, \dots, p$  is given by an array of D3 branes with  $n_i$  units of fundamental string charge dissolved on the  $i$ -th D3-brane.

In the second proposal [19, 20], a Wilson loop in the  $k$ -th anti-symmetric tensor representation (which is labeled by a Young-tableau of a column of  $k$  boxes) is given by a D5 brane with worldvolume  $AdS_2 \times S^4$  where  $k$  units of fundamental string charge are dissolved on the  $AdS_2$  worldvolume. Similarly to the first proposal a general representation with  $q$  columns with  $m_i$ ,  $i = 1, \dots, q$  boxes is given by an array of D5-branes with  $m_i$  units of fundamental string charge dissolved on the  $i$ -th D5-brane.

In both descriptions the D-branes are effectively treated as probes and the back-reaction on the geometry is neglected. The probe brane description of the Wilson loops is, however, very useful to determine the correct ansatz for the fluxes and scalars. The Born-Infeld form of the action for a Dp-brane is given by

$$S = \tau_p \int d^{p+1} \sigma e^{-\Phi} \sqrt{-\det(g + B + 2\pi\alpha' F)} + \mu_p \int C \wedge e^{2\pi\alpha' F} \tag{2.5}$$

A non-vanishing fundamental string charge manifests itself as a non-zero electro-magnetic field strength  $F_{\mu\nu}$  in the  $AdS_2$  worldvolume direction. In the probe approximation the worldvolume electric field will be a source for other supergravity fields.

For a probe D3-brane with  $AdS_2 \times S^2$  worldvolume the Born-Infeld action contains the following term linear in the electric field

$$2\pi\alpha'\tau_4 \int d^4\sigma e^{-\Phi} B_{\mu\nu} F^{\mu\nu} + 2\pi\alpha'\mu_4 \int F \wedge C_2 + \mu_4 \int C_4 \quad (2.6)$$

Consequently the electric field sources the NSNS two form potential  $B_2$  in the  $AdS_2$  directions and the RR two form potential  $C_2$  in the  $S^2$  direction.

For a probe D5-brane with  $AdS_2 \times S^4$  worldvolume the Born-Infeld action contains the following term linear in the electric field

$$2\pi\alpha'\tau_6 \int d^6\sigma e^{-\Phi} B_{\mu\nu} F^{\mu\nu} + \mu_6 \int C_6 + 2\pi\alpha'\mu_6 \int F \wedge C_4 \quad (2.7)$$

Consequently, the electric field sources the NSNS two form potential  $B_2$  in the  $AdS_2$  directions and the RR six form  $C_6$  potential along  $AdS_2 \times S^4$ . By electromagnetic duality the six form potential is related to the RR two form potential  $C_2$  which will be sourced along the  $S^2$  direction. Note that the strength of the probe brane sources differ in the two cases the direction in which the NSNS and RR fluxes are sourced are the same. In addition the probe brane sources the dilaton but not the RR axion.

In the rest of the paper the completely back-reacted solution which is dual to half-BPS Wilson loops will be constructed.

### 3. The Ansatz

As discussed in section 2, the half-BPS Wilson loop preserves the bosonic symmetries  $SO(2,1) \times SO(3) \times SO(5)$  and is invariant under 16 supersymmetries. Our conventions for Type IIB supergravity will follow the ones used in [10, 11] which coincide with those of [28]. As a Type IIB supergravity geometry,  $\Sigma$  will carry an orientation as well as a Riemannian metric, and is therefore a Riemann surface, generally non-compact, and with boundary. The symmetries determine the general form of the metric and the presence of the fundamental string charge and D5 branes determine the presence of NSNS and RR three form fluxes, leading to the following Ansatz for the supergravity fields.

#### 3.1 Ansatz for the type IIB fields

The appropriate supergravity ansatz for the metric is given by (2.4) which is a warped product of  $AdS_2 \times S^2 \times S^4$  factors over a two dimensional surface  $\Sigma$ .

$$ds^2 = f_1^2 ds_{AdS_2}^2 + f_2^2 ds_{S^2}^2 + f_4^2 ds_{S^4}^2 + ds_{\Sigma}^2 \quad (3.1)$$



where  $f_1, f_2, f_4$  and  $ds_{\Sigma}^2$  are real functions on  $\Sigma$ . We introduce an orthonormal frame,

$$\begin{array}{lll}
 AdS_2 & e^\mu = f_1 \hat{e}^\mu & \mu = 0, 1 \\
 S^2 & e^i = f_2 \hat{e}^i & i = 2, 3 \\
 S^4 & e^m = f_4 \hat{e}^m & m = 4, 5, 6, 7 \\
 \Sigma & e^a & a = 8, 9
 \end{array} \tag{3.2}$$

where  $\hat{e}^\mu, \hat{e}^i, \hat{e}^m$ , and  $e^a$  refer to orthonormal frames for the spaces  $AdS_2, S^2, S^4$ , and  $\Sigma$  respectively. In particular, we have<sup>1</sup>

$$\begin{aligned}
 ds_{AdS_2}^2 &= \eta_{\mu\nu} \hat{e}^\mu \otimes \hat{e}^\nu & ds_{S^2}^2 &= \delta_{ij} \hat{e}^i \otimes \hat{e}^j \\
 ds_{S^4}^2 &= \delta_{mn} \hat{e}^m \otimes \hat{e}^n & ds_{\Sigma}^2 &= \delta_{ab} e^a \otimes e^b
 \end{aligned} \tag{3.3}$$

where  $\eta = \text{diag}[-+]$ . The dilaton and axion fields are represented by the 1-forms  $P$  and  $Q$  which vary over  $\Sigma$ , and whose structure is given as follows,

$$P = p_a e^a, \quad Q = q_a e^a \tag{3.4}$$

while the anti-symmetric tensor forms  $F_{(5)}$  is self dual and given by

$$F_{(5)} = \left( - e^{0123} \wedge \mathcal{F} + e^{4567} \wedge *_2 \mathcal{F} \right) \tag{3.5}$$

In agreement with the symmetries and the probe analysis the three form field strength is constructed from the unit volume form on  $AdS_2$  and  $S^2$ ,

$$G = e^{01} \wedge \mathcal{G} + i e^{23} \wedge \mathcal{H} \tag{3.6}$$

where we have introduced the following 1-forms on  $\Sigma$  to represent the reduced fields,

$$\begin{aligned}
 \mathcal{G} &\equiv g_a e^a & \mathcal{F} &\equiv f_a e^a \\
 \mathcal{H} &\equiv h_a e^a & *_2 \mathcal{F} &\equiv \varepsilon^a_b f_a e^b & \varepsilon^{89} &= +1
 \end{aligned} \tag{3.7}$$

Here,  $f_a, q_a$  are real, while  $g_a, h_a, p_a$  are complex.

### 3.2 The general ten-dimensional Killing spinor

The requirement that 16 supersymmetries remain preserved by the Ansatz puts severe restrictions on the supergravity fields, which result from enforcing the BPS equations,  $\delta\lambda = \delta\psi_M = 0$ . Whenever the dilaton is subject to a non-trivial space-time variation,  $\partial_M \phi \neq 0$ , the dilatino BPS equation  $\delta\lambda = 0$  will allow for at most 16 independent supersymmetries  $\varepsilon$ . Therefore, the gravitino BPS equation cannot impose any further restrictions on the number of supersymmetries, but should instead simply give the space-time evolution of  $\varepsilon$ .

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<sup>1</sup>The convention of summation over repeated indices will be used throughout whenever no confusion is expected to arise, with the ranges of the various indices following the pattern of the frame in (3.2). Complex frame indices on  $\Sigma$  will often be used, with the following conventions,  $e^z = (e^8 + i e^9)/2$ ,  $e^{\bar{z}} = (e^8 - i e^9)/2$ , and the non-vanishing components of the metric on  $\Sigma$  are given by  $\delta_{z\bar{z}} = \delta_{\bar{z}z} = 2$ .

Thus, at any fixed point in the parameter space  $\Sigma$ ,  $\varepsilon$  must be a Killing spinor on each of the spheres  $S^2$  and  $S^4$ , as well as on  $AdS_2$ . The Killing spinor equations on  $AdS_2 \times S^2 \times S^4$  are given by

$$\begin{aligned} \left( \hat{\nabla}_\mu - \frac{1}{2} \eta_1 \gamma_\mu \otimes I_2 \otimes I_4 \right) \chi_{\eta_0}^{\eta_1, \eta_2, \eta_3} &= 0 & \mu = 0, 1 \\ \left( \hat{\nabla}_i - \frac{i}{2} \eta_2 I_2 \otimes \gamma_i \otimes I_4 \right) \chi_{\eta_0}^{\eta_1, \eta_2, \eta_3} &= 0 & i = 2, 3 \\ \left( \hat{\nabla}_m - \frac{i}{2} \eta_3 I_2 \otimes I_2 \otimes \gamma_m \right) \chi_{\eta_0}^{\eta_1, \eta_2, \eta_3} &= 0 & m = 4, 5, 6, 7 \end{aligned} \quad (3.8)$$

Here,  $\hat{\nabla}_\mu$ ,  $\hat{\nabla}_i$ , and  $\hat{\nabla}_m$ , are respectively the covariant derivatives acting in the Dirac spinor representations for  $AdS_2$ ,  $S^2$ , and  $S^4$ , with respect to the canonical spin connections associated with the frames  $\hat{e}^\mu$ ,  $\hat{e}^i$  and  $\hat{e}^m$ . The spinors  $\chi_{\eta_0}^{\eta_1, \eta_2, \eta_3}$  are 16-dimensional. The integrability conditions for each of these equations are automatically satisfied, so that for fixed  $\eta$ , the number of independent (complex) Killing spinors is respectively 2, 2 and 4 for the three equations. The indices  $\eta_0, \eta_1, \eta_2, \eta_3$  are independent and may take values  $\pm 1$ , and therefore uniquely label a basis of the 16-dimensional spinor space. The label  $\eta_0$  arises because the  $S^4$  equation is for a 4-component spinor, whose solutions are labeled by the pair  $(\eta_0, \eta_3)$ . Since the Killing spinor equation does not actually depend on  $\eta_0$ , this index simply labels two linearly independent spinors for which the reduced BPS equations are identical. Therefore, the index  $\eta_0$  will be dropped, with the understanding that the solution space for  $\chi^{\eta_1, \eta_2, \eta_3}$  remains 16-dimensional. (See [10] for the discussion of the analogous issues for  $AdS_4$ .)

For any one of the chirality matrices  $\gamma_{(s)}$ , for  $s = 1, 2, 3$ , the product  $\gamma_{(s)}\chi$  satisfies (3.8) with the opposite value of  $\eta_s$ . We may therefore identify the corresponding spinors,

$$\begin{aligned} \left( \gamma_{(1)} \otimes I_2 \otimes I_4 \right) \chi^{\eta_1, \eta_2, \eta_3} &= \chi^{-\eta_1, \eta_2, \eta_3} \\ \left( I_2 \otimes \gamma_{(2)} \otimes I_4 \right) \chi^{\eta_1, \eta_2, \eta_3} &= \chi^{\eta_1, -\eta_2, \eta_3} \\ \left( I_2 \otimes I_2 \otimes \gamma_{(3)} \right) \chi^{\eta_1, \eta_2, \eta_3} &= \chi^{\eta_1, \eta_2, -\eta_3} \end{aligned} \quad (3.9)$$

To examine the Killing spinor properties, we begin by decomposing the 32 component (complex) spinor  $\varepsilon$  onto the  $\Sigma$ -independent basis of spinors  $\chi_{\eta_0}^{\eta_1, \eta_2, \eta_3}$ , with coefficients which are  $\Sigma$ -dependent 2-component spinors  $\zeta_{\eta_1, \eta_2, \eta_3}$ ,

$$\varepsilon = \sum_{\eta_1, \eta_2, \eta_3} \chi^{\eta_1, \eta_2, \eta_3} \otimes \zeta_{\eta_1, \eta_2, \eta_3} \quad (3.10)$$

The 10-dimensional chirality condition  $\Gamma^{11}\varepsilon = -\varepsilon$  reduces to

$$\gamma_{(4)} \zeta_{-\eta_1, -\eta_2, -\eta_3} = -\zeta_{\eta_1, \eta_2, \eta_3} \quad (3.11)$$

The Killing spinor equations are invariant under charge conjugation  $\chi \rightarrow \chi^c$ , with

$$(\chi^c)^{\eta_1, \eta_2, \eta_3} = B_{(1)} \otimes B_{(2)} \otimes B_{(3)} (\chi^{\eta_1, \eta_2, \eta_3})^* \quad (3.12)$$

Since  $(B_{(1)} \otimes B_{(2)} \otimes B_{(3)})^* = B_{(1)} \otimes B_{(2)} \otimes B_{(3)}$ , and  $(B_{(1)} \otimes B_{(2)} \otimes B_{(3)})^2 = I_{16}$ , we may impose, without loss of generality, the reality condition  $\chi^c = \pm\chi$  on the basis. The sign assignments are related by (3.9), and after choosing  $\chi^{+++} = +B_{(1)} \otimes B_{(2)} \otimes B_{(3)}(\chi^{+++})^*$  are found to be

$$B_{(1)} \otimes B_{(2)} \otimes B_{(3)} (\chi^{\eta_1, \eta_2, \eta_3})^* = \eta_2 \chi^{\eta_1, \eta_2, \eta_3} \quad (3.13)$$

The  $\eta_2$  comes from the fact  $B_{(2)}$  anti-commutes with  $\gamma_{(2)}$ , while  $B_{(1)}$  and  $B_{(3)}$  commute with  $\gamma_{(1)}$  and  $\gamma_{(3)}$  respectively. Upon imposing the reality condition (3.13) on the basis of spinors  $\chi$ , and the chirality condition (3.11) on  $\zeta$ , and recalling that  $\chi^{\eta_1 \eta_2 \eta_3}$  has double degeneracy due to the suppressed quantum number  $\eta_0$ , we indeed recover 16 complex components for the spinor  $\varepsilon$ .

### 3.3 Notation

We introduce a matrix notation in the 8-dimensional space of  $\eta$  by,

$$\tau^{(ijk)} \equiv \tau^i \otimes \tau^j \otimes \tau^k \quad i, j, k = 0, 1, 2, 3 \quad (3.14)$$

where  $\tau^0 = I_2$ , and  $\tau^i$  with  $i = 1, 2, 3$  are the standard Pauli matrices. Multiplication by  $\tau^{(ijk)}$  is defined as follows,

$$(\tau^{(ijk)} \zeta)_{\eta_1, \eta_2, \eta_3} \equiv \sum_{\eta'_1, \eta'_2, \eta'_3} (\tau^i)_{\eta_1 \eta'_1} (\tau^j)_{\eta_2 \eta'_2} (\tau^k)_{\eta_3 \eta'_3} \zeta_{\eta'_1, \eta'_2, \eta'_3} \quad (3.15)$$

or more explicitly

$$\begin{aligned} \zeta_{\eta_1, \eta_2, \eta_3} &= (\zeta)_{\eta_1, \eta_2, \eta_3} \\ \zeta_{-\eta_1, \eta_2, \eta_3} &= (\tau^{(100)} \zeta)_{\eta_1, \eta_2, \eta_3} \\ \eta_1 \zeta_{-\eta_1, \eta_2, \eta_3} &= (+i \tau^{(200)} \zeta)_{\eta_1, \eta_2, \eta_3} \\ \eta_1 \zeta_{\eta_1, \eta_2, \eta_3} &= (\tau^{(300)} \zeta)_{\eta_1, \eta_2, \eta_3} \end{aligned} \quad (3.16)$$

This notation is analogous to the one introduced in [16], and used in [10].

## 4. Reduction of the BPS equations

The starting point is the supersymmetry transformation of the gravitino and dilatino

$$\begin{aligned} \delta\lambda &= i(\Gamma \cdot P) \mathcal{B}^{-1} \varepsilon^* - \frac{i}{24} (\Gamma \cdot G) \varepsilon \\ \delta\psi_M &= D_M \varepsilon + \frac{i}{480} (\Gamma \cdot F_{(5)}) \Gamma_M \varepsilon - \frac{1}{96} (\Gamma_M (\Gamma \cdot G) + 2(\Gamma \cdot G) \Gamma^M) \mathcal{B}^{-1} \varepsilon^* \end{aligned} \quad (4.1)$$

In order to preserve 16 supersymmetries, these equation must vanish for sixteen independent spinors  $\varepsilon$ . For non-constant dilaton ( $P \neq 0$ ), the dilatino equation will reduce the amount of supersymmetries from 32 to 16, and so the gravitino equation must impose no additional restrictions. To proceed, we first reduce these equations using the  $SO(2, 1) \times SO(3) \times SO(5)$  Ansatz. This will yield algebraic gravitino equations along the

directions of the maximally symmetric spaces,  $AdS_2$ ,  $S^2$ , and  $S^4$ . Using the decomposition of the ten-dimensional Killing spinor, (3.10), we are able to write the BPS equations entirely in terms of  $\zeta$ . The  $\chi^{\eta_1, \eta_2, \eta_3}$  then label the 16 independent supersymmetries. From the reduced BPS equations, we obtain a simple set of constraints on bilinears of  $\zeta$ . The net effect of these constraints is to impose a set of projections on  $\zeta$ . This has two benefits, first  $\zeta$  is reduced to two complex components and second using an explicit  $SL(2, \mathbf{R})$  transformation we map the problem to one with vanishing axion. That is real dilaton/axion field  $P$  and vanishing  $Q$ . This is similar to calculations in [10, 13], where the problem was also reduced to one with vanishing axion by an  $SL(2, \mathbf{R})$  transformation.

#### 4.1 The $SO(2, 1) \times SO(3) \times SO(5)$ reduction of the BPS equations

We give the explicit reduction of the dilatino equation. The gravitino equations can be reduced using the same method, so we simply quote the final result.

First we reduce the charge conjugate of the supersymmetry transformation spinor,

$$\mathcal{B}^{-1} \varepsilon^* \sum_{\eta_1, \eta_2, \eta_3} \chi^{\eta_1, \eta_2, \eta_3} \otimes (-i \eta_2) B_{(4)} \zeta_{\eta_1, -\eta_2, \eta_3}^* \quad (4.2)$$

where we have used the explicit form of  $\mathcal{B}$  given in (A.8), the reality condition on the basis of Killing spinors  $\chi^{\eta_1, \eta_2, \eta_3}$  given by (3.13), as well as the chirality condition given by (3.9). Using this result, the first term in the dilatino equation reduces to

$$i(\Gamma \cdot P) \mathcal{B}^{-1} \varepsilon^* - \sum_{\eta_1, \eta_2, \eta_3} \chi^{\eta_1, \eta_2, \eta_3} \otimes \left( p_a \sigma^a \sigma^2 \tau^{(131)} \zeta^* \right)_{\eta_1, \eta_2, \eta_3} \quad (4.3)$$

We have again used the matrix notation for  $\zeta$  introduced in (3.14). The  $\tau$ -matrices act on the eight-dimensional space spanned by the indices  $\{\eta_1, \eta_2, \eta_3\}$ , while the  $\sigma$ -matrices act on the two-component spinor,  $\zeta_{\eta_1, \eta_2, \eta_3}$ . We use a slight abuse of notation, and define  $\sigma^{8,9} \equiv \sigma^{1,2}$ . The second term reduces as

$$-\frac{i}{24} (\Gamma \cdot G) \varepsilon \sum_{\eta_1, \eta_2, \eta_3} \chi^{\eta_1, \eta_2, \eta_3} \otimes \left( +\frac{i}{4} g_a \sigma^a \tau^{(011)} \zeta - \frac{i}{4} h_a \sigma^a \tau^{(101)} \zeta \right)_{\eta_1, \eta_2, \eta_3} \quad (4.4)$$

Putting these two terms together yields the dilatino equation. Since  $\chi^{\eta_1, \eta_2, \eta_3}$  spans a basis, the dilatino equation requires that the coefficient of each  $\chi^{\eta_1, \eta_2, \eta_3}$  vanish separately. After dropping the summation and  $\chi^{\eta_1, \eta_2, \eta_3}$ , and multiplying by  $\tau^{(131)}$ , we have the final form of the reduced dilatino equation,

$$(d) \quad p_a \gamma^a \sigma^2 \zeta^* + \frac{1}{4} g_a \sigma^a \tau^{(120)} \zeta + \frac{i}{4} h_a \sigma^a \tau^{(030)} \zeta = 0 \quad (4.5)$$

The gravitino equations are obtained using the same methods. One additional step is to replace the covariant derivative of the spinor along the  $AdS_2$ ,  $S^2$ , and  $S^4$  directions by the corresponding group action,  $SO(2, 1)$ ,  $SO(3)$ , and  $SO(5)$  as defined in (3.8). It is important to note that an additional term appears in going from  $\nabla$  to  $\hat{\nabla}$ . This is due

to the warp factors appearing in the ten-dimensional metric. For example, the covariant derivative along  $AdS_2$  is given by

$$\nabla_\mu \varepsilon = \left( \frac{1}{f_1} \hat{\nabla}_\mu + \frac{1}{2} \frac{D_a f_1}{f_1} \Gamma_\mu \Gamma^a \right) \varepsilon \quad (4.6)$$

where  $D_a \equiv e_a^M \partial_M$  and  $M$  is a space-time (Einstein) index. After a bit of work, we obtain the following gravitino equations,

$$\begin{aligned} (\mu) \quad 0 &= -\frac{i}{2f_1} \tau^{(211)} \zeta + \frac{D_a f_1}{2f_1} \sigma^a \zeta + \frac{1}{2} f_a \sigma^a \tau^{(110)} \zeta + \frac{1}{16} \left( 3g_a \tau^{(120)} + i h_a \tau^{(030)} \right) \sigma^a \sigma^2 \zeta^* \\ (i) \quad 0 &= +\frac{1}{2f_2} \tau^{(021)} \zeta + \frac{D_a f_2}{2f_2} \sigma^a \zeta + \frac{1}{2} f_a \sigma^a \tau^{(110)} \zeta - \frac{1}{16} \left( g_a \tau^{(120)} + 3i h_a \tau^{(030)} \right) \sigma^a \sigma^2 \zeta^* \\ (m) \quad 0 &= +\frac{1}{2f_4} \tau^{(002)} \zeta + \frac{D_a f_4}{2f_4} \sigma^a \zeta - \frac{1}{2} f_a \sigma^a \tau^{(110)} \zeta - \frac{1}{16} \left( g_a \tau^{(120)} - i h_a \tau^{(030)} \right) \sigma^a \sigma^2 \zeta^* \\ (a) \quad 0 &= D_a \zeta + \frac{i}{2} \hat{\omega}_a \sigma^3 \zeta - \frac{i}{2} q_a \zeta + \frac{1}{2} f_b \sigma^b \sigma^a \tau^{(110)} \zeta + \frac{1}{16} \left( 3g_a - g_b \sigma^{ab} \right) \tau^{(120)} \sigma^2 \zeta^* \\ &\quad + \frac{i}{16} \left( -3h_a + h_b \sigma^{ab} \right) \tau^{(030)} \sigma^2 \zeta^* \end{aligned} \quad (4.7)$$

Here,  $\sigma^{ab}$  is defined by  $\sigma^{ab} \equiv \frac{1}{2}(\sigma^a \sigma^b - \sigma^b \sigma^a) = i \varepsilon^{ab} \sigma^3$ ; the derivatives  $D_a$  are defined with respect to the frame  $e^a$ , so that  $e^a D_a = d$ , the total differential on  $\Sigma$ .

## 4.2 Symmetries of the reduced BPS equations

The reduced BPS equations exhibit continuous as well as discrete symmetries, which will be exploited to further reduce the BPS equations.

### 4.2.1 Continuous symmetries

The continuous symmetries are as follows. Local frame rotations of the frame  $e^a$  on  $\Sigma$  generate a gauge symmetry  $U(1)_c$ , whose action on all fields is standard. The axion/dilaton field  $B$  transforms non-linearly under the continuous  $S$ -duality group  $SU(1,1)$  of Type IIB supergravity. As was discussed in section 3.1 of [10],  $B$  takes values in the coset  $SU(1,1)/U(1)_q$ , and  $SU(1,1)$  transformations on the fields are accompanied by local  $U(1)_q$  gauge transformations, given in section 3.1 of [10],

$$\begin{aligned} U(1)_q \quad \zeta &\rightarrow e^{i\theta/2} \zeta \\ q_a &\rightarrow q_a + D_a \theta & g_a &\rightarrow e^{i\theta} g_a \\ p_a &\rightarrow e^{2i\theta} p_a & h_a &\rightarrow e^{i\theta} h_a \end{aligned} \quad (4.8)$$

The real function  $\theta$  depends on the  $SU(1,1)$  transformation, as well as on the field  $B$ .

### 4.2.2 Discrete symmetries

The reduced BPS equations are invariant under three commuting involutions. The first two act on  $\zeta$  separately from  $\zeta^*$  and leave the fields  $f_a, p_a, q_a, g_a, h_a$  unchanged,

$$\begin{aligned} \mathcal{I} \zeta &= -\tau^{(111)} \sigma^3 \zeta \\ \mathcal{J} \zeta &= \tau^{(320)} \zeta \end{aligned} \quad (4.9)$$

Both  $\mathcal{I}$  and  $\mathcal{J}$  commute with the symmetries  $U(1)_q$  and  $U(1)_c$ .

### 4.2.3 Complex conjugation

The third involution  $\mathcal{K}$  amounts to complex conjugation. This operation acts non-trivially on all complex fields, and its action on  $\zeta$  depends on the basis of  $\tau$ -matrices. In a basis in which both  $\sigma^2$  and  $\tau^2$  are purely imaginary, the involution  $\mathcal{K}$  has the following form. Taking the complex conjugates of  $p_a, g_a, h_a$ , letting  $q_a \rightarrow -q_a$  and mapping  $\zeta \rightarrow i\tau^{(012)}\sigma^2\zeta^*$  will leave the BPS equations invariant.

Complex conjugation, defined this way, however, does not commute with the  $SU(1,1)$  transformations, since  $\zeta$  transforms under  $SU(1,1)$  by multiplication under a local  $U(1)_q$  gauge transformation. Therefore, we relax the previous definition of complex conjugation, and allow for complex conjugation modulo a  $U(1)_q$  gauge transformation with phase  $\theta$ ,

$$\begin{aligned} \mathcal{K}\zeta &= e^{i\theta}\tau^{(012)}\sigma^2\zeta^* & \mathcal{K}p_a &= e^{4i\theta}\bar{p}_a \\ \mathcal{K}q_a &= -q_a + 2D_a\theta & \mathcal{K}g_a &= e^{2i\theta}\bar{g}_a \\ & & \mathcal{K}h_a &= e^{2i\theta}\bar{h}_a \end{aligned} \tag{4.10}$$

which continues to be a symmetry of the BPS equations. The need for such a compensating gauge transformation should be clear from the fact that  $\zeta$  and  $\zeta^*$  transform with opposite phases under  $U(1)_q$ . On the other hand,  $\mathcal{K}$  commutes with the group  $U(1)_c$  of frame rotations.

In Type IIB theory only a single chirality is retained, so we have the condition

$$\mathcal{I}\zeta = -\tau^{(111)}\sigma^3\zeta = \zeta \tag{4.11}$$

This subspace is invariant under the remaining involutions, since  $\mathcal{J}$  and  $\mathcal{K}$  commute with  $\mathcal{I}$ .

## 5. Reality properties of the supersymmetric solutions

It is familiar from solving for the Janus solution with 4 supersymmetries in [13] and with 16 supersymmetries in [10] that the BPS equations imply certain reality conditions, which imply that every solution may be mapped into a “real” solution, for which the axion field vanishes. In [10], these reality conditions were derived by first obtaining from the BPS equations certain bilinear constraints on the spinors  $\zeta$ , and using those to show that only a single eigenspace of each involution  $\mathcal{I}$ ,  $\mathcal{J}$  and  $\mathcal{K}$  should be retained. The reality conditions for the present problem will be obtained in this manner as well. We repeat an abbreviated form of the analysis of [10] here, because, even though the analysis is very similar, its results will be different in subtle but crucial ways.

### 5.1 Restriction to a single eigenspace of $\mathcal{J}$

We shall show here that  $\zeta$  must obey the projection relation,

$$\mathcal{J}\zeta = \tau^{(320)}\zeta = \nu\zeta \tag{5.1}$$

where either the  $\nu = +1$  or the  $\nu = -1$  eigenspace is retained, but not both. To derive this result, we start by remarking that the chirality condition implies the vanishing of the

spinor bilinears  $\zeta^\dagger M \sigma^a \zeta = 0$ , for any  $\tau$ -matrix  $M$  which satisfies  $\{\tau^{(111)} \sigma^3, M \sigma^a\} = 0$ . Another set of bilinear constraints may be obtained by multiplying the dilatino equation by  $\zeta^\dagger M \sigma^{1,2}$ , where  $M$  is a  $\tau$ -matrix which satisfies

$$(T\tau^{(120)})^t = -T\tau^{(120)} \quad (T\tau^{(030)})^t = -T\tau^{(030)} \quad (5.2)$$

The  $g_a$  and  $h_a$  terms vanish and one is left with

$$\zeta^\dagger T \sigma^{1,2} \zeta = 0 \quad T \in \mathcal{T} \equiv \{\tau^{(200)}, \tau^{(201)}, \tau^{(310)}, \tau^{(311)}\} \quad (5.3)$$

We now move onto the gravitino equations. We first note that we have  $\tau^{(110)} \mathcal{T} = \mathcal{T}$ . It then follows that if we multiply the first three gravitino equations by  $\zeta^\dagger T \sigma^{0,3}$ , then only the first term in each equation survives and we obtain

$$\zeta^\dagger U \sigma^{0,3} \zeta = 0 \quad U \in \mathcal{U} \equiv \{\tau^{(011)}, \tau^{(010)}, \tau^{(101)}, \tau^{(100)}, \tau^{(221)}, \tau^{(220)}, \tau^{(331)}, \tau^{(330)}, \tau^{(202)}, \tau^{(203)}, \tau^{(312)}, \tau^{(313)}\} \quad (5.4)$$

These constraints can be solved by first finding a matrix which anti-commutes with all of the matrices in  $\mathcal{U}$ . There are two candidates  $\tau^{(320)}$  and  $\tau^{(231)}$ , which under multiplication by  $\tau^{(111)}$  are equivalent to each other. The  $U$  constraints are automatically satisfied upon imposing the projection condition (5.1); using the methods of appendix D of [10], one proves that the projection condition (5.1) is the general solution to the bilinear constraints.

## 5.2 Restriction to a single eigenspace of $\mathcal{K}$

We obtain additional constraints by multiplying the ( $\mu$ ), ( $i$ ), and ( $m$ ) gravitino equations by  $\zeta^\dagger \sigma^p M$ , where  $p = 0, 3$  so that either the  $g_a$  or  $h_a$  term vanishes, but not both. For the case when  $g_a$  survives, we choose  $M = \tau^{(002)}$ , and for the case when  $h_a$  survives, we choose  $M = \tau^{(003)}$ . We give explicit formulas for the first case, and simply quote the results from the second case. After multiplying the gravitino equations by  $2\zeta^\dagger \tau^{(002)} \sigma^p$ , we have

$$\begin{aligned} 0 &= -\frac{1}{f_1} \zeta^\dagger \tau^{(213)} \sigma^p \zeta + \frac{D_a f_1}{f_1} \zeta^\dagger \tau^{(002)} \sigma^p \sigma^a \zeta + f_a \zeta^\dagger \tau^{(112)} \sigma^p \sigma^a \zeta + \frac{3}{8} g_a \zeta^\dagger \tau^{(122)} \sigma^p \gamma^a \sigma^2 \zeta^* \\ 0 &= -\frac{i}{f_2} \zeta^\dagger \tau^{(023)} \sigma^p \zeta + \frac{D_a f_2}{f_2} \zeta^\dagger \tau^{(002)} \sigma^p \sigma^a \zeta + f_a \zeta^\dagger \tau^{(112)} \sigma^p \sigma^a \zeta - \frac{1}{8} g_a \zeta^\dagger \tau^{(122)} \sigma^p \gamma^a \sigma^2 \zeta^* \\ 0 &= +\frac{1}{f_4} \zeta^\dagger \tau^{(000)} \sigma^p \zeta + \frac{D_a f_4}{f_4} \zeta^\dagger \tau^{(002)} \sigma^p \sigma^a \zeta - f_a \zeta^\dagger \tau^{(112)} \sigma^p \sigma^a \zeta - \frac{1}{8} g_a \zeta^\dagger \tau^{(122)} \sigma^p \gamma^a \sigma^2 \zeta^* \end{aligned} \quad (5.5)$$

For  $p = 0$ , the first three terms are real in the first and third equations and so the imaginary part of the fourth term must vanish. For  $p = 3$  the first three terms are purely imaginary in the second equation, and again the real part of the fourth term must vanish. This gives two bilinear constraints involving  $g_a$ ; listing also the corresponding constraints involving  $h_a$ ,

$$\begin{aligned} \text{Im} \left( g_a \zeta^\dagger \tau^{(122)} \sigma^a \sigma^2 \zeta^* \right) &= 0 & \text{Re} \left( g_a \zeta^\dagger \tau^{(122)} \sigma^3 \sigma^a \sigma^2 \zeta^* \right) &= 0 \\ \text{Im} \left( i h_a \zeta^\dagger \tau^{(033)} \sigma^a \sigma^2 \zeta^* \right) &= 0 & \text{Re} \left( i h_a \zeta^\dagger \tau^{(033)} \sigma^3 \sigma^a \sigma^2 \zeta^* \right) &= 0 \end{aligned} \quad (5.6)$$

Taking  $p = 0$  in the second equation of (5.5), the last three terms are seen to be real using the above constraint, while the first term is purely imaginary, and so the first term must vanish. For the first and third equations of (5.5), we take  $p = 3$  and find that the first term is real while the last three terms are imaginary and so again, the first term must vanish. We quote the corresponding results for the  $h_a$  case

$$\begin{aligned}
 \zeta^\dagger \tau^{(022)} \zeta &= 0 & \zeta^\dagger \tau^{(023)} \zeta &= 0 \\
 \zeta^\dagger \tau^{(001)} \zeta &= 0 & \zeta^\dagger \tau^{(000)} \sigma^3 \zeta &= 0 \\
 \zeta^\dagger \tau^{(212)} \sigma^3 \zeta &= 0 & \zeta^\dagger \tau^{(213)} \sigma^3 \zeta &= 0
 \end{aligned} \tag{5.7}$$

The constraints (5.7) are solved by imposing a reality condition on  $\zeta$ ,

$$\sigma^2 \zeta^* = e^{-i\theta} \tau^{(012)} \zeta \tag{5.8}$$

where  $\theta$  is an arbitrary phase function on  $\Sigma$ , which is not fixed by the bilinear constraints. This result is readily verified by using (5.8) in the form  $\zeta^\dagger = e^{-i\theta} \zeta^t \sigma^2 \tau^{(012)}$  to eliminate  $\zeta^\dagger$  in (5.7) and then recognizing that the remaining equations are of the form  $\zeta^t M \zeta$  with  $M$  anti-symmetric, and thus vanishes. With the methods used in appendix D of [10], one demonstrates that (5.8) is in fact the most general solution to the bilinear constraints (5.7). The remaining constraints (5.6) may be simplified by eliminating  $\sigma^2 \zeta^*$ , from (5.6), using (5.8). Next, we use the assumption that  $\zeta^\dagger \tau^{(130)} \sigma^a \zeta$  and  $\zeta^\dagger \tau^{(021)} \sigma^a \zeta$  are not identically zero (this will be verified to hold on all the solutions) to obtain,

$$\text{Im} \left( p_a e^{-2i\theta} \right) = \text{Im} \left( i g_a e^{-i\theta} \right) \text{Im} \left( i h_a e^{-i\theta} \right) = 0 \tag{5.9}$$

Here we have included also the result of handling the dilatino equation.

### 5.3 SU(1,1) map to solutions with vanishing axion

The arguments that the reality conditions (5.9) imply that every solution to the BPS equations can be mapped to a “real” solution with vanishing axion proceed in parallel with the  $AdS_4$  case, treated in [10]. We repeat the key aspects here for completeness. The first equation in (5.9) implies that the dilaton/axion 1-form  $P$  satisfies  $P = e^{2i\theta} \tilde{P}$ , where  $\tilde{P}$  is a real form. Using the Bianchi identity  $dQ + iP \wedge \tilde{P} = 0$ , of eq (3.5) of [10], it follows that  $dQ = 0$ , so that  $Q$  is pure gauge. Additionally, from the SU(1,1) transformation laws (3.13) and (3.14), it follows that the phase  $\theta$  is to be interpreted as the accompanying  $U(1)_q$  gauge transformation of an SU(1,1) transformation that maps the solution to the BPS equations onto a solution for which  $\tilde{P}$  is real, and  $Q = 0$ . Performing now this SU(1,1) transformation on all fields, allows us to set  $e^{-i\theta} = i$ , so that the reality conditions (5.9) become,

$$\begin{aligned}
 \bar{p}_a &= p_a & \bar{g}_a &= g_a & a &= 8, 9 \\
 q_a &= 0 & \bar{h}_a &= h_a
 \end{aligned} \tag{5.10}$$

Complex conjugation is now a symmetry with  $\sigma^2 \zeta^* = i \tau^{(012)} \zeta$ .



## 5.4 Reduction to two dimensions

The projection conditions reduce the number of independent components of  $\zeta$  from sixteen complex components to two complex components. The next step is to make this reduction explicit in the BPS equations. The projection conditions are

$$\begin{aligned}\zeta &= -\tau^{(111)}\sigma^3\zeta \\ \zeta &= \nu\tau^{(320)}\zeta \\ \zeta^* &= i\sigma^2\tau^{(012)}\zeta\end{aligned}\tag{5.11}$$

In order to implement the first projection, it is convenient to use the following rotated basis for the  $\tau$ -matrices,<sup>2</sup>

$$\tau^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \tau^3 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\tag{5.12}$$

The reduction proceeds as follows. First we denote the components of  $\zeta$  by  $\zeta_{\eta_1\eta_2\eta_3\eta_4}$  where  $\eta_i = \pm$ . The first projection then fixes the overall sign of the  $\eta_i$  to be negative. The next step is to use the second projection constraint so that the BPS equations contain  $\tau$ -matrices whose action preserves the sign of  $\eta_1$ . This guarantees that equations with different  $\eta_1$  indices decouple, and we may fix  $\eta_1 = +$ . The equations with  $\eta_1 = -$  are then automatic. Using the new basis, the only change needed is to change the first term in the ( $\mu$ ) equation to

$$+\frac{i\nu}{2f_1}\tau^{(131)}\zeta\tag{5.13}$$

Next we use the reality condition to fix  $\eta_4 = -$ . In order to implement this last projection, we introduce a chiral basis for  $\zeta$ . We now retain the chiral component of each equation, and use the reality condition which relates  $\zeta_{\pm}$  as

$$\zeta_+ = -\tau^{(012)}\zeta^*\tag{5.14}$$

to eliminate  $\zeta_+$  from the equations. The equations become

$$\begin{aligned}(d) \quad 0 &= p_z\zeta_- + \frac{1}{4}g_z\tau^{(132)}\zeta_- - \frac{i}{4}h_z\tau^{(022)}\zeta_- \\ (\mu) \quad 0 &= +\frac{i\nu}{2f_1}\tau^{(123)}\zeta_-^* + \frac{D_z f_1}{2f_1}\zeta_- + \frac{1}{2}f_z\tau^{(110)}\zeta_- + \frac{3}{16}g_z\tau^{(132)}\zeta_- - \frac{i}{16}h_z\tau^{(022)}\zeta_- \\ (i) \quad 0 &= -\frac{1}{2f_2}\tau^{(033)}\zeta_-^* + \frac{D_z f_2}{2f_2}\zeta_- + \frac{1}{2}f_z\tau^{(110)}\zeta_- - \frac{1}{16}g_z\tau^{(132)}\zeta_- + \frac{3i}{16}h_z\tau^{(022)}\zeta_- \\ (m) \quad 0 &= -\frac{1}{2f_4}\tau^{(010)}\zeta_-^* + \frac{D_z f_4}{2f_4}\zeta_- - \frac{1}{2}f_z\tau^{(110)}\zeta_- - \frac{1}{16}g_z\tau^{(132)}\zeta_- - \frac{i}{16}h_z\tau^{(022)}\zeta_- \\ (a+) \quad 0 &= D_z\zeta_-^* + \frac{i}{2}\hat{\omega}_z\zeta_-^* + f_z\tau^{(110)}\zeta_-^* - \frac{1}{4}g_z\tau^{(132)}\zeta_-^* - \frac{i}{4}h_z\tau^{(022)}\zeta_-^* \\ (a-) \quad 0 &= D_z\zeta_- - \frac{i}{2}\hat{\omega}_z\zeta_- + \frac{1}{8}g_z\tau^{(132)}\zeta_- + \frac{i}{8}h_a\tau^{(022)}\zeta_-\end{aligned}\tag{5.15}$$

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<sup>2</sup>Notice that the transposition and complex conjugation properties of these matrices are identical to those in the standard basis.

These equations are now explicitly decoupled and we can restrict attention to the two-component spinors  $\xi$ , whose components we denote by  $\alpha$  and  $\beta$ ,

$$\zeta_- = \begin{pmatrix} \zeta_{++++} \\ \zeta_{+----} \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \equiv \xi \quad (5.16)$$

The key point is that the above  $\tau$ -matrices transform the above two components of  $\zeta$  denoted by  $\xi$  solely into each other. The action of the  $\tau$ -matrices on  $\xi$  takes the following form

$$\begin{aligned} \tau^{(132)} &= -\sigma^2 & \tau^{(123)} &= -\sigma^2 \\ \tau^{(110)} &= +\sigma^3 & \tau^{(033)} &= +\sigma^1 \\ \tau^{(022)} &= -\sigma^1 & \tau^{(010)} &= +\sigma^3 \end{aligned} \quad (5.17)$$

where  $\sigma$  are now in the standard basis of Pauli matrices. Using this notation, the reduced BPS equations become

$$\begin{aligned} (d) \quad 0 &= p_z \xi - \frac{1}{4} g_z \sigma^2 \xi + \frac{i}{4} h_z \sigma^1 \xi \\ (\mu) \quad 0 &= -\frac{i\nu}{2f_1} \sigma^2 \xi^* + \frac{1}{2} \frac{D_z f_1}{f_1} \xi + \frac{1}{2} f_z \sigma^3 \xi - \frac{3}{16} g_z \sigma^2 \xi + \frac{i}{16} h_z \sigma^1 \xi \\ (i) \quad 0 &= -\frac{1}{2f_2} \sigma^1 \xi^* + \frac{1}{2} \frac{D_z f_2}{f_2} \xi + \frac{1}{2} f_z \sigma^3 \xi + \frac{1}{16} g_z \sigma^2 \xi - \frac{3i}{16} h_z \sigma^1 \xi \\ (m) \quad 0 &= -\frac{1}{2f_4} \sigma^3 \xi^* + \frac{1}{2} \frac{D_z f_4}{f_4} \xi - \frac{1}{2} f_z \sigma^3 \xi + \frac{1}{16} g_z \sigma^2 \xi + \frac{i}{16} h_z \sigma^1 \xi \\ (a+) \quad 0 &= D_z \xi^* + \frac{i}{2} \hat{\omega}_z \xi^* + f_z \sigma^3 \xi^* + \frac{1}{4} g_z \sigma^2 \xi^* + \frac{i}{4} h_z \sigma^1 \xi^* \\ (a-) \quad 0 &= D_z \xi - \frac{i}{2} \hat{\omega}_z \xi - \frac{1}{8} g_z \sigma^2 \xi - \frac{i}{8} h_z \sigma^1 \xi \end{aligned} \quad (5.18)$$

It will be useful to have the transpose of the next to last equation as well,

$$0 = D_z \xi^\dagger + \frac{i}{2} \hat{\omega}_z \xi^\dagger + f_z \xi^\dagger \sigma^3 - \frac{1}{4} g_z \xi^\dagger \sigma^2 + \frac{i}{4} h_z \xi^\dagger \sigma^1 \quad (5.19)$$

### 5.5 Algebraic relations for the radii $f_1$ , $f_2$ , and $f_4$

The reduced gravitino BPS equations contain sectors which are purely algebraic and may be used to produce algebraic expressions for the radii  $f_1$ ,  $f_2$  and  $f_4$  in terms of  $\xi$ , or equivalently, in terms of  $\alpha$  and  $\beta$ . The results are as follows,

$$\begin{aligned} f_1 &= -\xi^\dagger \xi = -(\bar{\alpha}\alpha + \bar{\beta}\beta) \\ f_2 &= \nu \xi^\dagger \sigma^3 \xi = \nu(\bar{\alpha}\alpha - \bar{\beta}\beta) \\ f_4 &= \nu \xi^\dagger \sigma^1 \xi = \nu(\bar{\alpha}\beta + \bar{\beta}\alpha) \end{aligned} \quad (5.20)$$

The derivation is completely analogous to the derivation given in [10] for the corresponding radii, and will not be reproduced here. The key in the derivation is to multiply the  $(\mu)$ ,  $(i)$ ,

and  $(m)$  equations respectively by  $\xi^\dagger$ ,  $\xi^\dagger\sigma^3$ , and  $\xi^\dagger\sigma^1$ , and use  $(a+)$  and  $(a-)$  equations to derive the following derivative equations,

$$D_z \left( \xi^\dagger \sigma^p \xi \right) = -f_z \xi^\dagger \sigma^3 \sigma^p \xi + \frac{1}{4} \xi^\dagger (g_z \sigma^2 - i h_z \sigma^1) \sigma^p \xi + \frac{1}{8} \xi^\dagger \sigma^p (g_z \sigma^2 + i h_z \sigma^1) \xi \quad (5.21)$$

for  $p = 0, 1, 3$ . Eliminating  $g_z$ ,  $h_z$ , and  $f_z$  from both sets of equations yields (5.20) up to overall multiplicative constants, which are fixed by using the remaining linearly independent combinations of the  $(\mu)$ ,  $(i)$ , and  $(m)$  equations.

### 5.6 The remaining reduced BPS equations

Once the relations for the radii (5.20) have been extracted from the reduced BPS equations (5.18), only equations  $(d)$ ,  $(a\pm)$  and one particular combination of the  $(m)$  equations,

$$(m) \quad 0 = -i\nu - i f_z \xi^t \sigma^1 \xi + \frac{1}{8} g_z \xi^t \xi + \frac{1}{8} h_z \xi^t \sigma^3 \xi \quad (5.22)$$

remain. We may choose conformal coordinates  $w, \bar{w}$  on  $\Sigma$ , in terms of which the metric on  $\Sigma$  takes the form,  $ds_\Sigma^2 = 4\rho^2 dw d\bar{w}$ . The frames, derivatives and connection are then,

$$\begin{aligned} e^z &= \rho dw & D_z &= \rho^{-1} \partial_w & \hat{\omega}_z &= +i\rho^{-2} \partial_w \rho \\ e^{\bar{z}} &= \rho d\bar{w} & D_{\bar{z}} &= \rho^{-1} \partial_{\bar{w}} & \hat{\omega}_{\bar{z}} &= -i\rho^{-2} \partial_{\bar{w}} \rho \end{aligned} \quad (5.23)$$

In local complex coordinates, and expressing  $\xi$  in terms of  $\alpha$  and  $\beta$  using (5.16), the remaining reduced BPS equations (5.22) take the form,

$$\begin{aligned} 4p_z \alpha + i(g_z + h_z) \beta &= 0 & (d) \\ 4p_z \beta - i(g_z - h_z) \alpha &= 0 \\ -i\nu - 2i\alpha\beta f_z + \frac{1}{8}(g_z + h_z)\alpha^2 + \frac{1}{8}(g_z - h_z)\beta^2 &= 0 & (m) \\ \frac{1}{\rho} \partial_w \bar{\alpha} - \frac{1}{2\rho^2} (\partial_w \rho) \bar{\alpha} + f_z \bar{\alpha} - \frac{i}{4} (g_z - h_z) \bar{\beta} &= 0 & (a+) \\ \frac{1}{\rho} \partial_w \bar{\beta} - \frac{1}{2\rho^2} (\partial_w \rho) \bar{\beta} - f_z \bar{\beta} + \frac{i}{4} (g_z + h_z) \bar{\alpha} &= 0 \\ \frac{1}{\rho} \partial_w \alpha + \frac{1}{2\rho^2} (\partial_w \rho) \alpha + \frac{i}{8} (g_z - h_z) \beta &= 0 & (a-) \\ \frac{1}{\rho} \partial_w \beta + \frac{1}{2\rho^2} (\partial_w \rho) \beta - \frac{i}{8} (g_z + h_z) \alpha &= 0 \end{aligned} \quad (5.24)$$

These equations will be the starting point for the complete solution of the reduced BPS equations, to be carried out in the subsequent sections.

## 6. The BPS equations form an integrable system

The exact solution of the  $AdS_4$  BPS equations in [10] was obtained by mapping the BPS equations onto an integrable system which was then mapped onto free field equations in turn. The same method also works for the problem at hand. The integrable system is very

similar, but not identical, to the one found and used in [10]. The differences between the two system will produce key differences between the physical supergravity solutions. For this reason, and for the sake of completeness, we shall reproduce here the key manipulations required for the integrable system.

First, the dilatino equation (*d*) may be used to solve for  $g_z$  and  $h_z$  in terms of  $\alpha$ ,  $\beta$ ,  $p_z$ ,

$$\begin{aligned} g_z &= 2i \left( \frac{\alpha}{\beta} - \frac{\beta}{\alpha} \right) p_z \\ h_z &= 2i \left( \frac{\alpha}{\beta} + \frac{\beta}{\alpha} \right) p_z \end{aligned} \quad (6.1)$$

in all equations, and the (*m*) equation may be solved to obtain  $f_z$ ,

$$f_z = -\frac{\nu}{2\alpha\beta} + \frac{\alpha^4 - \beta^4}{4\alpha^2\beta^2} p_z \quad (6.2)$$

The remaining equations may be cast in the following form,

$$\begin{aligned} \partial_w \ln \left( \frac{\bar{\alpha}}{\bar{\beta}} \right) - \frac{\nu\rho}{\alpha\beta} + \frac{\alpha^4 - \beta^4}{2\alpha^2\beta^2} \rho p_z - \left( \frac{|\beta|^2}{|\alpha|^2} - \frac{|\alpha|^2}{|\beta|^2} \right) \rho p_z &= 0 \quad (a+) \\ \partial_w \ln (\bar{\alpha}\bar{\beta}) - \partial_w \ln \rho - \left( \frac{|\beta|^2}{|\alpha|^2} + \frac{|\alpha|^2}{|\beta|^2} \right) \rho p_z &= 0 \\ \partial_w \alpha + \frac{1}{2\rho} (\partial_w \rho) \alpha + \frac{1}{2} \frac{\beta^2}{\alpha} \rho p_z &= 0 \quad (a-) \\ \partial_w \beta + \frac{1}{2\rho} (\partial_w \rho) \beta + \frac{1}{2} \frac{\alpha^2}{\beta} \rho p_z &= 0 \end{aligned} \quad (6.3)$$

Notice that, while there is considerable similarity with the reduced BPS equations (7.3) and (7.4) of [10], the details differ. In particular, the coefficients of  $\rho p_z$  are qualitatively different.

### 6.1 Solution of the (*a*-) system

Multiplying the first (*a*-) equation of (6.3) by  $2\rho\alpha$  and the second by  $2\rho\beta$ , we obtain the equivalent equations,

$$\begin{aligned} \partial_w (\rho\alpha^2) + (\rho p_z) \rho\beta^2 &= 0 \\ \partial_w (\rho\beta^2) + (\rho p_z) \rho\alpha^2 &= 0 \end{aligned} \quad (6.4)$$

It follows that  $\rho p_z$  is the gradient of a real function, which we shall denote by  $\phi$ . The dilaton field, in standard normalization, is related to  $\phi$  by  $\phi = \Phi/2$ ,

$$\rho p_z = \partial_w \phi \quad (6.5)$$

Adding and subtracting both equations in (6.4), we get

$$\begin{aligned} \partial_w (\ln (\rho(\alpha^2 + \beta^2)) + \phi) &= 0 \\ \partial_w (\ln (\rho(\alpha^2 - \beta^2)) - \phi) &= 0 \end{aligned} \quad (6.6)$$

These equations may be solved in terms of two arbitrary holomorphic functions  $\kappa$  and  $\lambda$ ,

$$\begin{aligned}\rho(\alpha^2 + \beta^2) &= \bar{\kappa} e^{-\bar{\lambda} - \phi} \\ \rho(\alpha^2 - \beta^2) &= \bar{\kappa} e^{+\bar{\lambda} + \phi}\end{aligned}\tag{6.7}$$

To be more precise,  $\lambda$  is a scalar function, but  $\kappa$  is a form of weight  $(1, 0)$ , in a normalization where the frame component  $e^z$  is a form of weight  $(-1, 0)$ . This gives a complete solution of the  $(a-)$  system. The product of the two relations in (6.7) gives,

$$\rho^2 (\alpha^4 - \beta^4) = \bar{\kappa}^2\tag{6.8}$$

The parametrization in terms of  $\kappa$  and  $\lambda$  is convenient since the following will occur,

$$\begin{aligned}\frac{\alpha^2}{\beta^2} &= \frac{1 + e^{2\phi + 2\bar{\lambda}}}{1 - e^{2\phi + 2\bar{\lambda}}} \\ 4\rho^2 \alpha^2 \beta^2 &= \bar{\kappa}^2 \left( e^{-2\phi - 2\bar{\lambda}} - e^{2\phi + 2\bar{\lambda}} \right)\end{aligned}\tag{6.9}$$

The spinor components  $\alpha$  and  $\beta$  may be computed as well,

$$\begin{aligned}\alpha &= (\bar{\kappa}/\rho)^{-\frac{1}{2}} \text{ch}(\phi + \bar{\lambda})^{\frac{1}{2}} \\ \beta &= i(\bar{\kappa}/\rho)^{-\frac{1}{2}} \text{sh}(\phi + \bar{\lambda})^{\frac{1}{2}}\end{aligned}\tag{6.10}$$

but this of course has required a choice of signs, which we fix by the above formula.

## 6.2 Solution of the $(a+)$ system

Taking the logarithmic derivatives of the complex conjugates of equations (6.9) provides the combinations of derivatives applied to  $\bar{\alpha}$  and  $\bar{\beta}$  that enter into the  $(a+)$  equations,

$$\begin{aligned}\partial_w \ln \left( \frac{\bar{\alpha}}{\bar{\beta}} \right) &= \frac{1}{2} \left( \frac{\bar{\alpha}^2}{\bar{\beta}^2} - \frac{\bar{\beta}^2}{\bar{\alpha}^2} \right) (\partial_w \phi + \partial_w \lambda) \\ \partial_w \ln (\rho \bar{\alpha} \bar{\beta}) &= \partial_w \ln \kappa - \frac{1}{2} \left( \frac{\bar{\alpha}^2}{\bar{\beta}^2} + \frac{\bar{\beta}^2}{\bar{\alpha}^2} \right) (\partial_w \phi + \partial_w \lambda)\end{aligned}\tag{6.11}$$

Eliminating now the logarithmic derivatives between (6.11) and the  $(a+)$  equations, and then eliminating any further algebraic  $\alpha$ - and  $\beta$ -dependences through (6.7) and (6.9), we obtain the following first order system,

$$\left[ \frac{1}{\text{sh}(2\phi + 2\lambda)} + \frac{1}{\text{sh}(2\phi + 2\bar{\lambda})} - \frac{2 \text{ch}(\lambda - \bar{\lambda})}{|\text{sh}(2\phi + 2\lambda)|} \right] \partial_w \phi = \frac{i\sqrt{2\nu}\rho^2\bar{\kappa}^{-1}}{\text{sh}(2\phi + 2\bar{\lambda})^{\frac{1}{2}}} - \frac{\partial_w \lambda}{\text{sh}(2\phi + 2\lambda)}\tag{6.12}$$

and

$$\partial_w \ln \rho^2 = \partial_w \ln \kappa + \frac{\text{ch}(2\phi + 2\lambda)}{\text{sh}(2\phi + 2\lambda)} (\partial_w \phi + \partial_w \lambda) - 2 \frac{\text{ch}(2\phi + \lambda + \bar{\lambda})}{|\text{sh}(2\phi + 2\lambda)|} \partial_w \phi\tag{6.13}$$

This system of first order equations is virtually identical to the one encountered in equation (7.12) of [10]; the differences are by a replacement of  $\text{sh}$  by  $\text{ch}$  in the numerator of the third term on the lhs of the first equation, and in the numerator of the second term on the rhs in the second equation, in addition to various signs and factors of  $i$ .

### 6.3 Integrability and the universal dilaton equation

The above system of first order differential equations is integrable. The proof is completely analogous to the proof given for the corresponding first order system in [10]. Here, we shall limit ourselves to stressing the minor differences. A key ingredient in the proof was the derivation of a second order partial equation for the dilaton  $\phi$  and the holomorphic function  $\lambda$  alone. This is also the case here, and the corresponding equations is obtained by eliminating  $\rho$  between the above first order equations,

$$\partial_w \partial_{\bar{w}} \phi - 2 \frac{\text{sh}(4\phi + 2\lambda + 2\bar{\lambda})}{|\text{sh}(2\phi + 2\lambda)|^2} \partial_w \phi \partial_{\bar{w}} \phi + \frac{\partial_{\bar{w}} \phi \partial_w \lambda}{\text{sh}(\lambda - \bar{\lambda})} \frac{\text{sh}(2\phi + 2\bar{\lambda})}{\text{sh}(2\phi + 2\lambda)} \left( -\frac{1}{\text{ch}(\lambda - \bar{\lambda})} - \frac{\text{sh}(2\phi + 2\lambda)^{\frac{1}{2}}}{\text{sh}(2\phi + 2\bar{\lambda})^{\frac{1}{2}}} \right) + \text{c.c.} \quad (6.14)$$

The integrability conditions on  $\phi$  and  $\ln \rho^2$  now follows as in [10]. Note the minor differences between (6.14) and the corresponding equation (7.16) in [10].

## 7. Complete analytical solution

In [10], a judiciously chosen change of variables was discovered, under which the dilaton equation, analogous to (6.14), was mapped to an equation akin to Liouville and Sine-Gordon field theory, and the first order system, analogous to (6.12), is mapped to the corresponding Bäcklund pair. This integrable system, in turn, was mapped onto a set of linear equations, to which the full solution may be derived and expressed in terms of two real harmonic functions. With minor, but significant, alterations, these changes of variables may be adapted to the present system of equations, which may also be solved completely via these methods.

The change of variables (which coincides with the one carried out in [10] for the dilaton),

$$e^{2i\vartheta} \equiv \frac{\text{sh}(2\phi + 2\lambda)}{\text{sh}(2\phi + 2\bar{\lambda})} \quad (7.1)$$

maps the dilaton equation (6.14) to the following equation,

$$\partial_{\bar{w}} \partial_w \vartheta + \frac{1}{\sin \mu} \left( e^{-i\vartheta} \partial_{\bar{w}} \vartheta \partial_w \mu + e^{i\vartheta} \partial_w \vartheta \partial_{\bar{w}} \mu \right) - 2 \frac{\cos \mu}{(\sin \mu)^2} \partial_w \lambda \partial_{\bar{w}} \bar{\lambda} \sin \vartheta = 0 \quad (7.2)$$

where we use the notation  $\lambda - \bar{\lambda} = i\mu$ , with  $\mu$  real. An alternative form of the equation exploits the special relation that exists between the second and third terms to recast (7.2) as a conservation equation,

$$\partial_{\bar{w}} \left( \partial_w \vartheta + 2i \frac{\partial_w \mu}{\sin \mu} e^{-i\vartheta} \right) + \partial_w \left( \partial_{\bar{w}} \vartheta - 2i \frac{\partial_{\bar{w}} \mu}{\sin \mu} e^{+i\vartheta} \right) = 0 \quad (7.3)$$

Equations (7.2) and (7.3) are clearly again of the Liouville and Sine-Gordon type, but differ in subtle ways from the corresponding equation found in [10].

### 7.1 Changing variables in the first order system

To simplify the first order system (6.12), we carry out the change of variables (7.1) but, in addition, need to use instead of  $\rho$  the new variable  $\hat{\rho}$ , defined by,

$$\rho^8 \equiv \frac{\hat{\rho}^8}{16} \kappa^4 \bar{\kappa}^4 (\sin 2\mu)^2 \frac{\cos \mu - \cos \vartheta}{(\cos \mu + \cos \vartheta)^3} \quad (7.4)$$

The first-order system of equations becomes,

$$\begin{aligned} \partial_w \vartheta &= -i \frac{\partial_w \mu}{\sin \mu} e^{-i\vartheta} - i \partial_w \ln \sin \mu + i\nu \hat{\rho}^2 \kappa e^{i\vartheta/2} \\ \partial_w \ln \hat{\rho}^2 &= \frac{i}{2} \partial_w \vartheta - \frac{\partial_w \mu}{\sin \mu} e^{-i\vartheta} \end{aligned} \quad (7.5)$$

We make a further change of variables, and define

$$\psi \equiv \frac{\sin \mu}{\hat{\rho}^2} e^{-i\vartheta/2} \quad (7.6)$$

It is immediate to recast the first order system in terms of  $\Psi$ ,

$$\begin{aligned} \partial_w \psi &= \nu \kappa \sin \mu \\ \partial_w \bar{\psi} &= \bar{\psi} \frac{\cos \mu}{\sin \mu} \partial_w \mu + \psi \frac{\partial_w \mu}{\sin \mu} \end{aligned} \quad (7.7)$$

Integrability of this system is now easily checked.

### 7.2 Solving the first order system

We begin by defining the holomorphic scalar functions  $\mathcal{A}(w)$  and  $\mathcal{B}(w)$  by,

$$\begin{aligned} \partial_w \mathcal{A} &= -\frac{\nu}{2} \kappa e^{+\lambda} \\ \partial_w \mathcal{B} &= -\frac{\nu}{2} \kappa e^{-\lambda} \end{aligned} \quad (7.8)$$

up to additive constants. The first equation in (7.7) is readily solved, and we find,

$$\psi(w, \bar{w}) = ie^{-\bar{\lambda}}(\bar{w})\mathcal{A}(w) - ie^{\bar{\lambda}}(\bar{w})\mathcal{B}(w) + \overline{\varphi(w)} \quad (7.9)$$

where  $\varphi(w)$  is an as yet to be determined holomorphic function. Next, we substitute this result into the second equation of (7.7), and find,

$$2e^{-\bar{\lambda}}\bar{\mathcal{A}} - 2e^{\bar{\lambda}}\bar{\mathcal{B}} - 2e^{-\bar{\lambda}}\mathcal{A} + 2e^{\bar{\lambda}}\mathcal{B} = i \left( e^{\lambda-\bar{\lambda}} - e^{-\lambda+\bar{\lambda}} \right) \frac{\partial_w \varphi}{\partial_w \lambda} - i \left( e^{\lambda-\bar{\lambda}} + e^{-\lambda+\bar{\lambda}} \right) \varphi - 2i\bar{\varphi} \quad (7.10)$$

By the same arguments as we used in [10], the general solution is found to be

$$\varphi = -ie^{-\lambda}(\mathcal{A} - r_1) + ie^{\lambda}(\mathcal{B} + r_2) \quad (7.11)$$

where  $r_1, r_2$  are two arbitrary real constants. Assembling now all contributions to  $\psi$ , we find,

$$\psi = ie^{-\bar{\lambda}}(\mathcal{A} + \bar{\mathcal{A}}) - ie^{\bar{\lambda}}(\mathcal{B} + \bar{\mathcal{B}}) \quad (7.12)$$

and the constants  $r_1, r_2$  have been absorbed into the integration constants of  $\mathcal{A}$  and  $\mathcal{B}$  without loss of generality.

### 7.3 Solving for the dilaton

From the definition of the complex field  $\psi$  in (7.6), it is manifest that both  $\vartheta$  and  $\hat{\rho}$  may be recovered from  $\psi$ . In turn, from  $\vartheta$  and  $\hat{\rho}$ , one derives the  $\Sigma$ -metric  $\rho$  using (7.4), and the dilaton  $\phi$  using (7.1), and one derives  $\alpha$  and  $\beta$  using (6.10), and ultimately the metric factors  $f_1$ ,  $f_2$ , and  $f_4$  using (5.20). The results are most conveniently expressed in terms of two real harmonic functions  $h_1$  and  $h_2$  (on  $\Sigma$ ), which are defined by,

$$\begin{aligned} h_1 &\equiv \mathcal{A} + \bar{\mathcal{A}} \\ h_2 &\equiv \mathcal{B} + \bar{\mathcal{B}} \end{aligned} \tag{7.13}$$

These steps are all familiar from [10]. In terms of  $h_1$  and  $h_2$ , we have

$$\begin{aligned} \psi &= ie^{-\bar{\lambda}}h_1 - ie^{\lambda}h_2 \\ \kappa^2 &= 4\partial_w h_1 \partial_w h_2 \\ e^{2\lambda} &= \frac{\partial_w h_1}{\partial_w h_2} \end{aligned} \tag{7.14}$$

The function  $\vartheta$  is then given by

$$e^{i\vartheta} = -\frac{e^{-\lambda}h_1 - e^{\lambda}h_2}{e^{-\bar{\lambda}}h_1 - e^{\bar{\lambda}}h_2} \tag{7.15}$$

The dilaton may be computed using (7.1), and the third formula in (7.14), and we find,

$$e^{4\phi} = -\frac{2h_1h_2|\partial_w h_2|^2 - h_2^2(\partial_w h_1 \partial_{\bar{w}} h_2 + \partial_w h_2 \partial_{\bar{w}} h_1)}{2h_1h_2|\partial_w h_1|^2 - h_1^2(\partial_w h_1 \partial_{\bar{w}} h_2 + \partial_w h_2 \partial_{\bar{w}} h_1)} \tag{7.16}$$

Notice the overall sign difference with the  $AdS_4$  dilaton solution of [10]. In view of the positivity requirements on  $e^{4\phi}$ , this sign difference will have drastic effects on the singularity behavior of the solutions.

A number of combinations of the harmonic functions will be pervasive, and we shall give them shorthand notations,

$$\begin{aligned} V &= \partial_u h_1 \partial_{\bar{u}} h_2 - \partial_{\bar{u}} h_1 \partial_u h_2 \\ W &= \partial_u h_1 \partial_{\bar{u}} h_2 + \partial_{\bar{u}} h_1 \partial_u h_2 \\ N_1 &= 2h_1h_2|\partial_u h_1|^2 - h_1^2 W \\ N_2 &= 2h_1h_2|\partial_u h_2|^2 - h_2^2 W \end{aligned} \tag{7.17}$$

The dilaton and metric formulas will take simpler forms, and the regularity conditions on the solutions will be naturally expressed in terms  $V, W, N_1$ , and  $N_2$ .

### 7.4 Solving for the $\Sigma$ -metric

Combining the definition of  $\psi$  in (7.6), the solution for  $\psi$  in (7.14), the formula for  $\vartheta$  in (7.15), and the relation between  $\rho$  and  $\hat{\rho}$  in (7.4), and expressing the result directly in terms of the notation (7.17), we find,

$$\rho^8 = -\frac{W^2 N_1 N_2}{h_1^4 h_2^4} \tag{7.18}$$



This equation may be further simplified by including a factor of the dilaton. Multiplying by  $e^{\pm 2\phi}$  we obtain a perfect square on the right hand side. Taking the square root, we have

$$e^{+2\phi} \rho^4 = \frac{|WN_2|}{h_1^2 h_2^2} \qquad e^{-2\phi} \rho^4 = \frac{|WN_1|}{h_1^2 h_2^2} \qquad (7.19)$$

Notice that positivity of  $e^{4\phi}$  and of  $\rho^8$  require the same condition that  $N_1 N_2 < 0$ .

### 7.5 Solving for the radii

The radii of  $AdS_2$ ,  $S^2$  and  $S^4$  are given respectively by (5.20), namely

$$\begin{aligned} f_1 &= -(\alpha^* \alpha + \beta^* \beta) \\ f_2 &= \nu(\alpha^* \alpha - \beta^* \beta) \\ f_4 &= \nu(\alpha^* \beta + \beta^* \alpha) \end{aligned} \qquad (7.20)$$

It is important to note that the metric factors are explicitly real. Furthermore,  $f_1$  is never zero unless  $\alpha$  and  $\beta$  both vanish; in this case all of the metric factors would be zero. The solution (6.10) for  $\alpha$  and  $\beta$  in terms of  $\lambda$  and  $\kappa$ , gives us formulas for  $\alpha$  and  $\beta$ , and thus for  $f_1$ ,  $f_2$  and  $f_3$ . It is somewhat more convenient to work with products of metric factors multiplied by the  $\Sigma$ -metric factor  $\rho$ ,

$$\rho^2 f_1 f_2 = -2\nu W \qquad (7.21)$$

$$\begin{aligned} \rho^2 f_1 f_4 &= -2\nu \left( (\partial_w h_2)^2 ((\partial_{\bar{w}} h_2)^2 e^{-4\phi} - (\partial_{\bar{w}} h_1)^2) \right)^{\frac{1}{2}} + \text{c.c.} \\ \rho^2 f_2 f_4 &= 2 \left( (\partial_w h_1)^2 ((\partial_{\bar{w}} h_2)^2 - e^{4\phi} (\partial_{\bar{w}} h_1)^2) \right)^{\frac{1}{2}} + \text{c.c.} \end{aligned} \qquad (7.22)$$

Using the dilaton solution (7.16), the terms under the square roots simplify because some of their factors are now perfect squares. After some simplifications, one obtains,

$$\begin{aligned} \rho^2 f_1 f_4 &= -2\nu \left( \frac{W}{N_2} \right)^{\frac{1}{2}} \left( \partial_w h_2 (h_1 \partial_{\bar{w}} h_2 - h_2 \partial_{\bar{w}} h_1) + s \partial_{\bar{w}} h_2 (h_1 \partial_w h_2 - h_2 \partial_w h_1) \right) \\ \rho^2 f_2 f_4 &= 2 \left( \frac{-W}{N_1} \right)^{\frac{1}{2}} \left( \partial_w h_1 (h_1 \partial_{\bar{w}} h_2 - h_2 \partial_{\bar{w}} h_1) + s \partial_{\bar{w}} h_1 (h_1 \partial_w h_2 - h_2 \partial_w h_1) \right) \end{aligned} \qquad (7.23)$$

Here, we have introduced the common sign factor  $s = \pm 1$ , which arises from taking the square roots. Reality of  $\rho^2 f_1 f_4$  and  $\rho^2 f_2 f_4$  forces a correlation between the sign  $s$  in the above parentheses and the signs of  $W, N_1, N_2$ . Recall that to have a well-defined dilaton, we need  $N_1$  and  $N_2$  to be of opposite signs. If  $WN_1 < 0$ , then the square roots in both expressions are taken of positive quantities, which requires the  $s = +1$ ; while if  $WN_1 > 0$ , the square root is taken over a negative quantity, which requires  $s = -1$ .

With definite sign choices, the formulas simplify considerably, and for the sake of further simplification, we shall present here their squares,

$$\begin{aligned} N_1 W < 0 \qquad \rho^4 f_1^2 f_2^2 &= +4W^2 \\ \rho^4 f_1^2 f_4^2 &= +4N_2 W h_2^{-2} \\ \rho^4 f_2^2 f_4^2 &= -4N_1 W h_1^{-2} \end{aligned} \qquad (7.24)$$

and for the opposite signs,

$$\begin{aligned}
 N_1 W > 0 \quad \rho^4 f_1^2 f_2^2 &= +4W^2 \\
 \rho^4 f_1^2 f_4^2 &= +4h_2^2 W V^2 N_2^{-1} \\
 \rho^4 f_2^2 f_4^2 &= -4h_1^2 W V^2 N_1^{-1}
 \end{aligned} \tag{7.25}$$

From these combinations, we extract the radii themselves, as well as the products of pairs of radii, which will be very useful later on,

$$\begin{aligned}
 N_1 W < 0 \quad f_1^4 &= -4e^{+2\phi} h_1^4 \frac{W}{N_1} & f_2^2 f_4^2 &= 4e^{-2\phi} h_2^2 \\
 f_2^4 &= +4e^{-2\phi} h_2^4 \frac{W}{N_2} & f_1^2 f_4^2 &= 4e^{+2\phi} h_1^2 \\
 f_4^4 &= +4e^{-2\phi} \frac{N_2}{W} & f_1^2 f_2^2 &= 4e^{+2\phi} \frac{h_1^2 h_2^2 W}{N_2}
 \end{aligned} \tag{7.26}$$

and for the opposite signs,

$$\begin{aligned}
 N_1 W > 0 \quad f_1^4 &= -4e^{-2\phi} h_2^4 \frac{W}{N_2} & f_2^2 f_4^2 &= -4e^{-2\phi} h_1^4 h_2^2 \frac{V^2}{N_1^2} \\
 f_2^4 &= +4e^{+2\phi} h_1^4 \frac{W}{N_1} & f_1^2 f_4^2 &= -4e^{+2\phi} h_1^2 h_2^4 \frac{V^2}{N_2^2} \\
 f_4^4 &= +4e^{+2\phi} h_1^4 h_2^4 \frac{V^4}{N_1 N_2^2 W} & f_1^2 f_2^2 &= +4e^{-2\phi} \frac{h_1^2 h_2^2 W}{N_1}
 \end{aligned} \tag{7.27}$$

Recall that, since  $V$  is purely imaginary, we have  $V^2 < 0$ .

### 7.6 The three form fluxes and the Bianchi identities

It will be helpful to have explicit expressions for the complex gauge potential  $B_{(2)}$ , and the associated complex field strength,

$$F_{(3)} = dB_{(2)} = H_{(3)} + iC_{(3)} \tag{7.28}$$

Its real part  $H_{(3)}$  is the NSNS 3-form field strength, while its imaginary part  $C_{(3)}$  is the RR 3-form field strength. The Bianchi identities on the 3-forms are just the statement that  $F_{(3)}$  is closed. These forms will be needed to evaluate the corresponding charges. In terms of the  $AdS_2 \times S^2 \times S^4 \times \Sigma$  Ansatz, and conformal gauge on  $\Sigma$ , the forms reduce as follows,

$$\begin{aligned}
 H_{(3)} &= \hat{e}^{01} \wedge e^{+\phi} (g_z \rho dw + g_{\bar{z}} \rho d\bar{w}) \\
 C_{(3)} &= \hat{e}^{23} \wedge e^{-\phi} (h_z \rho dw + h_{\bar{z}} \rho d\bar{w})
 \end{aligned} \tag{7.29}$$

The forms  $\hat{e}^{01}$  and  $\hat{e}^{23}$  are the unit volume forms respectively on  $AdS_2$  and on  $S^2$  and are automatically closed. The requirement that  $F_{(3)}$  is closed then implies the local existence of real functions  $b_1$  and  $b_2$ , such that

$$\begin{aligned}
 B_{(2)} &= b_1 \hat{e}^{01} + i b_2 \hat{e}^{23} \\
 db_1 &= e^{+\phi} (g_z \rho dw + g_{\bar{z}} \rho d\bar{w}) \\
 db_2 &= e^{-\phi} (h_z \rho dw + h_{\bar{z}} \rho d\bar{w})
 \end{aligned} \tag{7.30}$$

The calculation of  $b_{1,2}$  proceeds in analogy with the corresponding calculation in [10]. The starting point is obtained by recasting the weight (1, 0) part of the 1-forms  $db_{1,2}$  in terms of the dilaton and  $\alpha$  and  $\beta$ ,

$$\begin{aligned} e^{+\phi} f_1^2 \rho g_z &= 2ie^{+\phi}(\alpha^2 - \beta^2) \bar{\alpha} \bar{\beta} \left( \frac{\alpha \bar{\alpha}}{\beta \bar{\beta}} + \frac{\beta \bar{\beta}}{\alpha \bar{\alpha}} + 2 \right) \partial_w \phi \\ e^{-\phi} f_2^2 \rho h_z &= 2ie^{-\phi}(\alpha^2 + \beta^2) \bar{\alpha} \bar{\beta} \left( \frac{\alpha \bar{\alpha}}{\beta \bar{\beta}} + \frac{\beta \bar{\beta}}{\alpha \bar{\alpha}} - 2 \right) \partial_w \phi \end{aligned} \quad (7.31)$$

and expressing these forms as total derivatives. The second ( $a+$ ) equation in (6.3) is used to express the combination of the first two terms in the large parenthesis on the rhs in terms of a total derivative. Using also the complex conjugates of the solutions in (6.9) to the ( $a-$ ) equations, the entire combinations may be expressed as derivatives,

$$\begin{aligned} e^{+\phi} f_1^2 \rho g_z &= \partial_w \left( 2i\rho^{-1} \bar{\alpha} \bar{\beta} \bar{\kappa} e^{+2\phi+\bar{\lambda}} \right) \\ e^{-\phi} f_2^2 \rho h_z &= \partial_w \left( 2i\rho^{-1} \bar{\alpha} \bar{\beta} \bar{\kappa} e^{-2\phi-\bar{\lambda}} \right) \end{aligned} \quad (7.32)$$

We may now extract  $b_{1,2}$  up to some anti-holomorphic functions as

$$\begin{aligned} b_1 &= 2i\rho^{-1} \bar{\alpha} \bar{\beta} \bar{\kappa} e^{+2\phi+\bar{\lambda}} + \eta_1(\bar{w}) \\ b_2 &= 2i\rho^{-1} \bar{\alpha} \bar{\beta} \bar{\kappa} e^{-2\phi-\bar{\lambda}} + \eta_2(\bar{w}) \end{aligned} \quad (7.33)$$

Since  $b_{1,2}$  must be real, the following combinations must be harmonic,

$$2i\rho^{-1} \bar{\alpha} \bar{\beta} \bar{\kappa} e^{\pm(2\phi+\bar{\lambda})} + 2i\rho^{-1} \alpha \beta \bar{\kappa} e^{\pm(2\phi+\lambda)} \quad (7.34)$$

and this may indeed be checked explicitly to be the case, in parallel with [10]. The remaining calculation may be carried out along the lines of [10] as well. We introduce the duals of  $h_1$  and  $h_2$  denoted by  $\tilde{h}_1$  and  $\tilde{h}_2$ , so that

$$\begin{aligned} b_1 &= -2i \frac{h_1^2 h_2 V}{N_1} - 2\tilde{h}_2 & \tilde{h}_1 &\equiv i(\mathcal{A} - \bar{\mathcal{A}}) \\ b_2 &= -2i \frac{h_1 h_2^2 V}{N_2} + 2\tilde{h}_1 & \tilde{h}_2 &\equiv i(\mathcal{B} - \bar{\mathcal{B}}) \end{aligned} \quad (7.35)$$

Note that the overall sign of the 2-form  $B_{(2)}$  and its 3-form field strength  $F_{(3)}$  depend on the sign choice made for  $\alpha$  and  $\beta$  in taking the square roots is (6.10).

### 7.7 Transformation rules

There are a number of simple transformations on the harmonic functions  $h_1$  and  $h_2$  which result in simple transformation laws for the dilaton, metric factors, and 3-form field  $g_z, h_z$ .

- (i) Common scaling  $h_{1,2} \rightarrow c^2 h_{1,2}$ ; leaves the dilaton  $\phi$  invariant and

$$\begin{aligned} \rho &\rightarrow c \rho f_1 \rightarrow c f_1 & f_2 &\rightarrow c f_2 & f_3 &\rightarrow c f_3^* \\ g_z &\rightarrow c^{-1} g_z & h_z &\rightarrow c^{-1} h_z & f_z &\rightarrow c^{-1} f_z \end{aligned} \quad (7.36)$$

(ii) Inverse scaling  $h_1 \rightarrow e^{-\phi_0} h_1$  and  $h_2 \rightarrow e^{\phi_0} h_2$ ; shifts the dilaton,  $\phi \rightarrow \phi + \phi_0$  and leaves all other fields invariant.

(iii) Sign reversal  $h_1 \rightarrow -h_1$ ,  $h_2 \rightarrow h_2$ ,  $\nu \rightarrow -\nu$ ; leaves the dilaton and  $\rho$  invariant, and

$$\begin{aligned} W &\rightarrow -W & N_{1,2} &\rightarrow -N_{1,2} \\ f_1 &\rightarrow f_1 & f_2 &\rightarrow -f_2 & f_4 &\rightarrow f_4 \\ g_z &\rightarrow -g_z & h_z &\rightarrow h_z & f_z &\rightarrow -f_z \end{aligned} \tag{7.37}$$

1. Interchange  $h_1 \leftrightarrow h_2$ ;

$$\begin{aligned} W &\rightarrow W & N_{1,2} &\rightarrow N_{2,1} & \phi &\rightarrow -\phi \\ f_1 &\rightarrow f_1 & f_2 &\rightarrow f_2 & f_4 &\rightarrow f'_4 \\ g_z &\rightarrow ih_z & h_z &\rightarrow ig_z & f_z &\rightarrow f'_z \end{aligned} \tag{7.38}$$

The main effect of this transformation is to swap the  $S^2$  with  $AdS_2$ , which effectively swaps electric charge with magnetic charge. The prime indicates that the fields do not transform simply. For example for the metric factor  $f_4$ , the exchange of  $h_1$  and  $h_2$  changes the choice of sign for the square root in (7.23).

(iv) Lastly, we give the action of the ‘‘S-duality’’ transformation coming from the  $SU(1, 1)$  symmetry of Type IIB supergravity. It’s action is given by setting  $\theta = \frac{\pi}{4}$  in the  $U(1)_q$  transformation,

$$V : \quad e^{+\phi} \rightarrow e^{-\phi} \quad g_z \rightarrow ig_z \quad h_z \rightarrow ih_z \tag{7.39}$$

The effect here, in contrast to transformation of item 4, is to exchange the NSNS and RR fields.

## 8. Constant dilaton solutions and $AdS_5 \times S^5$

Constancy of the dilaton,  $\phi = \phi_0$ , implies that the solution is  $AdS_5 \times S^5$  with  $SO(2, 4) \times SO(6)$ -invariant metric. To show this, it is most convenient to analyze the first order equation (6.12), under the assumption  $\partial_w \phi_0 = 0$ . This readily gives an expression for the  $\Sigma$ -metric  $\rho$ , in terms of the constant  $\phi_0$  and the holomorphic functions  $\kappa$  and  $\lambda$ ,

$$\sqrt{2}\nu\rho^2 = -i\bar{\kappa}\partial_w\lambda \frac{\text{sh}(2\phi_0 + 2\bar{\lambda})^{\frac{1}{2}}}{\text{sh}(2\phi_0 + 2\lambda)} \tag{8.1}$$

The second first-order equation (6.13) is then automatically solved. Reality of  $\nu\rho^2$  implies the following relation between  $\kappa$ ,  $\lambda$  and the constant  $\phi_0$ ,

$$-i\frac{\partial_w\lambda}{\kappa\text{sh}(2\phi_0 + 2\lambda)^{\frac{3}{2}}} = i\frac{\partial_{\bar{w}}\bar{\lambda}}{\bar{\kappa}\text{sh}(2\phi_0 + 2\bar{\lambda})^{\frac{3}{2}}} \tag{8.2}$$

Since the left hand side is holomorphic, and the right hand side is anti-holomorphic, both must be a constant and real. For later convenience, we shall denote this constant by

$1/(2\sqrt{2}\nu c^2)$ , where  $c$  is real. This equation gives  $\kappa$  as a function of  $\lambda$  and the constant  $c$ . It also gives a formula for  $\rho^2$ , and this the  $\Sigma$ -metric, in terms of  $\lambda$  only,

$$ds_{\Sigma}^2 = \rho^2 |dw|^2 = \frac{2c^2 |d\lambda|^2}{|\text{sh}(2\phi_0 + 2\lambda)|^2} \tag{8.3}$$

where we have used the fact that  $dw\partial_w\lambda = d\lambda$ , since  $\lambda$  is holomorphic. In view of the positivity of  $\rho^2$ , we must choose  $c$  real. It is natural to change conformal coordinates on  $\Sigma$  and use directly the coordinate in which  $ds_{\Sigma}^2$  is flat Euclidean,

$$ds_{\Sigma}^2 = 2c^2 |dz|^2 \qquad z = -\frac{i\pi}{2} + \frac{1}{2} \ln \left( \frac{\text{sh}(\phi_0 + \lambda)}{\text{ch}(\phi_0 + \lambda)} \right) \tag{8.4}$$

The shift of  $z$  by  $-i\pi/2$  is made for later convenience. Using (7.8), and the above expressions for  $\kappa$  and  $\lambda$ , now all expressed as functions of the new conformal coordinate  $z$ , we readily obtain the combinations  $\partial_z\mathcal{A}$  and  $\partial_z\mathcal{B}$ , and hence the harmonic functions  $h_1$  and  $h_2$ ,

$$\begin{aligned} \partial_z\mathcal{A} &= 2c^2 e^{-\phi_0} \text{sh}(z) & h_1 &= c^2 e^{-\phi_0} \left( e^z + e^{-z} + e^{\bar{z}} + e^{-\bar{z}} \right) \\ \partial_z\mathcal{B} &= 2c^2 e^{+\phi_0} \text{ch}(z) & h_2 &= c^2 e^{+\phi_0} \left( e^z - e^{-z} + e^{\bar{z}} - e^{-\bar{z}} \right) \end{aligned} \tag{8.5}$$

In the sequel of this section, we shall show that this geometry indeed uniquely corresponds to the  $AdS_5 \times S^5$  solution.

### 8.1 The $AdS_5 \times S^5$ solution

At this point, it is useful to express the real harmonic functions  $h_1$  and  $h_2$  in terms of the real coordinates  $x, y$  defined by  $z = x + iy$ . From (8.5), we then have

$$\begin{aligned} h_1 &= 4c^2 e^{-\phi_0} (\text{ch } x) \cos y \\ h_2 &= 4c^2 e^{+\phi_0} (\text{sh } x) \cos y \end{aligned} \tag{8.6}$$

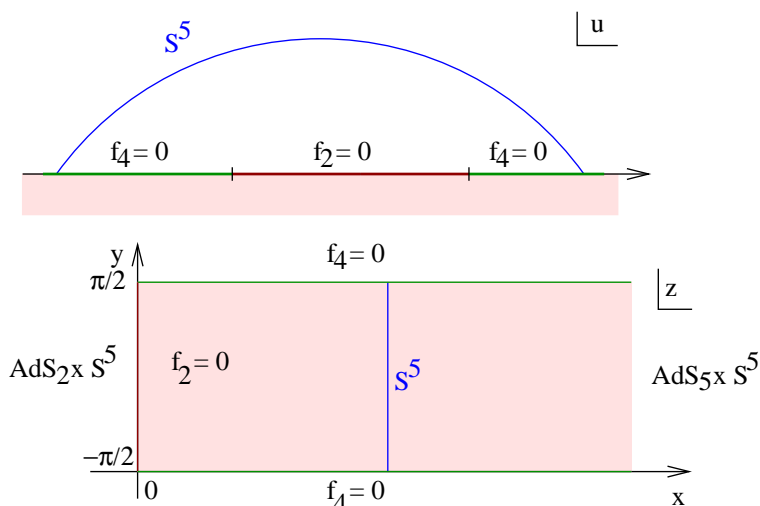
We first compute the composite functions defined in (7.17),

$$\begin{aligned} W &= +4c^4 \text{sh } 2x \\ V &= -4i c^4 \sin 2y \\ N_1 &= -64c^8 e^{-2\phi_0} (\cos y)^4 \text{sh } 2x \\ N_2 &= +64c^8 e^{+2\phi_0} (\cos y)^4 \text{sh } 2x \end{aligned} \tag{8.7}$$

The dilaton and  $\rho$  metric factor are given by

$$e^{4\phi} = e^{4\phi_0} \qquad \rho^4 = 4c^4 \tag{8.8}$$

To evaluate the remaining metric factors, we note that  $W$  and  $N_1$  have opposite signs, and so we must use the first set of metric equations, (7.26). The metric factors are readily



**Figure 2:** The  $u$ -plane (top) and  $z$ -plane (bottom) for  $AdS_5 \times S^5$ .

computed and found to be given by the customary expressions in a coordinate system where the  $\Sigma$ -metric  $\rho$  is constant,

$$\begin{aligned} f_1^2 &= 8c^2(\text{ch } x)^2 \\ f_2^2 &= 8c^2(\text{sh } x)^2 \\ f_4^2 &= 8c^2(\cos y)^2 \end{aligned} \tag{8.9}$$

This is  $AdS_5 \times S_5$  with radius  $R^2 = 8c^2$ . The boundary structure of the space is as follows,

- The domain is the half-strip,  $0 \leq x \leq \infty$  with  $-\pi/2 \leq y \leq \pi/2$  (see figure 2);
- $AdS_2$  never shrinks to zero;
- $S^2$  shrinks to zero when  $W$  and  $h_2$  are zero at  $x = 0$ ;
- $S^4$  shrinks to zero when  $V$ ,  $h_1$  and  $h_2$  are zero at  $y = -\pi/2, \pi/2$ ;
- Throughout the inside of the domain, we have  $h_1 > 0$  and  $h_2 > 0$ ;
- $(-N_1)$  and  $N_2$  are positive, which guarantees a real dilaton.

### 8.2 Mapping the $AdS_5 \times S^5$ solution to the lower half-plane

To generalize the  $AdS_5 \times S^5$  solution, it will be more convenient to parametrize the solution by the lower half-plane, as in [10]. The half-strip  $0 \leq \text{Re}(w) < \infty$  and  $-\frac{\pi}{2} \leq \text{Im}(w) \leq \frac{\pi}{2}$  is mapped to the lower half-plane  $\text{Im}(v) < 0$  by the following conformal transformation,

$$v = -i \text{sh}(z) \tag{8.10}$$

In particular, the imaginary segment from  $[-\pi i/2, +\pi i/2]$  in the  $z$ -strip is mapped to the real segment from  $[-1, +1]$  in the  $v$ -plane. The half-line segments  $-i\pi/2 + \mathbf{R}^+$  and

$+i\pi/2 + \mathbf{R}^+$  in the  $z$ -strip are mapped respectively to the real axis segments  $[-1, -\infty]$  and  $[+1, +\infty]$  of the  $v$ -plane. Finally, the real segment  $[0, +\infty]$  in the  $z$ -strip is mapped to the negative imaginary axis in the  $v$ -plane.

To simplify the expressions, we make use of transformations 1. and 2. of subsection 7.7 to fix  $c = 1/\sqrt{2}$  and  $\phi_0 = 0$ , without loss of generality. Expressing the harmonic functions (8.5) in terms of the  $v$ -coordinates, we obtain,

$$\begin{aligned} h_1 &= \sqrt{1 - v^2} + \text{c.c.} \\ h_2 &= i(v - \bar{v}) \end{aligned} \tag{8.11}$$

Note that  $h_1$  vanishes along the real axis when  $|v| > 1$  and is real when  $|v| < 1$ . We shall choose the square root such that  $h_1 > 0$  for  $|v| < 1$ . The function  $h_2$  is manifestly positive for  $\text{Im}(v) < 0$  and vanishes along the real axis. The differentials of  $h_1$  and  $h_2$  are given by,

$$\begin{aligned} \partial h_1 &= -\frac{v dv}{\sqrt{1 - v^2}} \\ \partial h_2 &= i dv \end{aligned} \tag{8.12}$$

We see that in the  $AdS_5 \times S^5$  solution, the differentials are both constant multiples of  $dv$  as  $v \rightarrow \infty$ . This results in a divergence in the harmonic functions as  $v \rightarrow \infty$ .

It is natural to investigate whether the  $AdS_5 \times S^5$  solution admits simple generalizations of the type achieved by the supersymmetric Janus solution in [10]. Upon maintaining the same boundary conditions as for  $AdS_5 \times S^5$ , and the same branch points and cuts, the only possible generalization is,

$$\partial h_1 = -\frac{(v - a)dv}{\sqrt{1 - v^2}} \tag{8.13}$$

while keeping  $h_2$  unchanged, for some constant  $a$ . The function  $h_1$  is readily obtained by integrating the differential  $\partial h_1$ , and we have,,

$$h_1 = b + \sqrt{1 - v^2} - a \arcsin(v) + \text{c.c.} \tag{8.14}$$

Requiring  $h_1$  to vanish at  $v = \pm 1$  leads to  $a = b = 0$ , and the solution is just  $AdS_5 \times S^5$ .

### 8.3 Preparing for generalization

In the formulation of the  $AdS_5 \times S^5$  differentials given in (8.12), there are branch points at  $v = \pm 1$ , and a double pole at  $\infty$ . Notice that  $\infty$  is not a branch point in this formulation. Actually, there are two different points at  $\infty$ , one for each Riemann sheet. To put this solution in a form which is closer to the Janus solution and hyperelliptic Ansatz of [11], we perform a Möbius transformation on the coordinate  $v$  to a coordinate  $u$ , in which one of the branch points, say  $v = -1$ , is sent to  $u = \infty$ , and  $v = \infty$  is brought back to 2 distinct finite points. The Möbius transformation in question is  $u - u_0 = -1/(v + 1)$ , and

the resulting differentials are<sup>3</sup>

$$\begin{aligned}\partial h_1 &= -i \frac{(u - u_0 + 1) du}{(u - u_0)^2 \sqrt{u - e_1}} \\ \partial h_2 &= i \frac{du}{(u - u_0)^2}\end{aligned}\tag{8.15}$$

where the branch point is now at  $e_1 = u_0 - 1/2$ . The lower  $v$ -half-plane is mapped into the lower  $u$ -half-plane (see figure 2). The two infinities in the  $v$ -half-plane are mapped to the points which we denote by  $(u_0, +s(u_0))$  and  $(u_0, -s(u_0))$ , where  $s^2(u) = u - e_1$ .

The formulation (8.15) of  $AdS_5 \times S^5$  is better suited to exhibit the similarities and differences between the  $AdS_2$  solution of this paper and the  $AdS_4$  solution of paper [11]. For  $AdS_4$ , the differentials  $\partial h_1$  and  $\partial h_2$  had double poles at the branch points, but nowhere else. For  $AdS_2$ , the differentials  $\partial h_1$  and  $\partial h_2$  are regular at the branch points (in fact  $\partial h_2$  vanishes at both branch points), but now there are two double poles away from the branch points. It is the formulation of (8.15) that will be most readily amenable to generalization.

## 9. Conditions for regular solutions

The general solution to the BPS equations, obtained in section 7 in terms of two real harmonic functions  $h_1$  and  $h_2$  on  $\Sigma$ , does not always correspond to acceptable Type IIB geometries. For example, the harmonic functions must be chosen so that the right hand side of the dilaton equation (7.16) yields a positive or zero value for  $e^\phi$ . In addition, the metric factors may develop singularities. For example, the  $AdS_5 \times S^5$  metric factors in (8.9) diverge as  $x \rightarrow \infty$ . After mapping  $\infty$  to a finite point  $u_0$  located on  $\partial\Sigma$ , this divergence shows up as a double pole in the Abelian differentials given in (8.15). The gravity dual to a Wilson loop should have only one such asymptotic  $AdS_5 \times S^5$  region. Correspondingly, we will allow only one such  $AdS_5 \times S^5$  singularity located on  $\partial\Sigma$ .

We restrict attention to solutions where  $N_1$ ,  $N_2$  and  $W$  have definite sign throughout the domain  $\Sigma$ . If this were not the case, these quantities would develop zeros on the inside of  $\Sigma$ , which would — generically — lead to singularities in the dilaton, or in the radii  $f_1$ ,  $f_2$  or  $f_4$ . Without loss of generality, transformation 3. of section 7.7 may be used to choose  $W$  positive. Reality of the dilaton  $\phi$  and positivity of  $\rho^2$  then require that  $N_1$  and  $N_2$  have opposite signs. We now make the additional assumption that  $WN_1 < 0$ , so that the expressions for the metric factors in (7.26) are valid. We repeat them here for convinence

$$\begin{aligned}f_1^2 f_4^2 &= 4e^{2\phi} h_1^2 \\ f_2^2 f_4^2 &= 4e^{-2\phi} h_2^2 \\ f_1^4 &= -e^{2\phi} h_1^4 \frac{W}{N_1}\end{aligned}\tag{9.1}$$

Using transformation 4., it is possible to map solutions with  $WN_1 > 0$  to solutions with  $WN_1 < 0$ , but this map does not guarantee that regular solutions are mapped to regular solutions. In particular, the problem of generating regular solutions in the case  $WN_1 > 0$  is an open question.

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<sup>3</sup>Here, we omit a factor of  $1/\sqrt{2}$  in  $\partial h_1$ , which may be restored at will using transformations 1. and 2.



### 9.1 Topology conditions

Since we are interested in the gravity duals of Wilson loops, we require the ten-dimensional geometry to have a boundary with the topology of  $AdS_5 \times S^5$ . This means that every point in  $\partial\Sigma$ , except for the  $AdS_5 \times S^5$  singularity mentioned in the first paragraph, must correspond to a regular interior point in the ten-dimensional geometry. This is achieved by requiring either  $S^2$  or  $S^4$  to shrink to zero size on  $\partial\Sigma$ . The product  $f_2^2 f_4^2$  given in (9.1) vanishes if and only if  $h_2$  vanishes (assuming a regular dilaton). This means the boundary is specified completely by  $h_2 = 0$ ,

$$\partial\Sigma = \{\text{Im}(v) < 0 \text{ such that } h_2(v) = 0\} \tag{9.2}$$

We make the additional requirement that the product  $f_2^2 f_4^2$  vanishes on a single line so that  $\partial\Sigma$  has a single component. It is useful to choose conformal coordinates on the lower half-plane in which  $h_2 = -2\text{Im}(v)$ ; since  $h_2$  is a harmonic function to begin with, this can always be done. These are the coordinates used in (8.11) for  $AdS_5 \times S^5$ . Note that  $h_2$  is positive throughout the lower half-plane. The product  $f_2^2 f_4^2$  vanishes automatically on the real axis, which is now the boundary,  $\partial\Sigma$ .

Using the restrictions of the first three paragraphs of this section, namely the  $AdS_5 \times S^5$  singularity restriction from the first paragraph, the sign choices of the second paragraph, and the boundary  $AdS_5 \times S^5$  topological restrictions of the third paragraph, we derive a set of restrictions on the harmonic functions. We first compute a few useful quantities in the well-adapted coordinates  $v = x + iy$ , and enforce the sign choices,

$$\begin{aligned} W &= -\partial_y h_1 \geq 0 \\ N_1 &= -h_1 \left( y(\partial_x h_1)^2 + y(\partial_y h_1)^2 + h_1 W \right) \leq 0 \\ N_2 &= -4y \left( h_1 - yW \right) \geq 0 \end{aligned} \tag{9.3}$$

The metric factor  $f_1$  never vanishes. This follows from (7.20), where it is obvious that  $f_1$  may vanish if and only if  $\alpha$  and  $\beta$  vanish. The metric products in (9.1) imply the harmonic functions  $h_1$  and  $h_2$  may not contain singularities, except for the  $AdS_5 \times S^5$  singularity located on  $\partial\Sigma$ . Finally, the product for  $f_1 f_4$  given in (9.1) implies that  $f_4$  vanishes whenever  $h_1$  vanishes, which may only happen on  $\partial\Sigma$ . Given the sign choice  $W > 0$  and the explicit form in (9.3) it follows that  $h_1$  is positive throughout the lower half-plane.

### 9.2 Summary of topology and regularity conditions

We summarize the restrictions as follows.

- (R1) The harmonic functions  $h_1$  and  $h_2$  are non-singular except for one point on  $\partial\Sigma$  corresponding to the asymptotic  $AdS_5 \times S^5$  region.
- (R2) The boundary  $\partial\Sigma$  is defined by the line  $h_2 = 0$ .
- (R3) The function  $h_1$  may vanish only on the segment of  $\partial\Sigma$  where  $S^4$  shrinking to zero. The regions of  $\partial\Sigma$  where  $h_1 \neq 0$  correspond to  $S^2$  shrinking to zero.

**(R4)** The functions  $h_1$  and  $h_2$  are positive definite inside  $\Sigma$ , but may vanish on  $\partial\Sigma$ .

Given that  $h_1$  vanishes only on  $\partial\Sigma$ , it follows from (9.3) that  $N_2$  is a sum of two positive quantities and vanishes only on  $\partial\Sigma$ . In order for the dilaton to be regular,  $N_1$  may vanish only when  $N_2$  does, and so only vanishes on  $\partial\Sigma$ . Finally, in order for  $f_1$ , given in (9.1), to be regular and non-zero in the bulk of  $\Sigma$ ,  $W$  may vanish only when  $N_1$  vanishes, and so only vanishes on  $\partial\Sigma$  as well. Given that  $h_1, h_2, W, N_1$  and  $N_2$  never vanish in the bulk of  $\Sigma$ , it can be verified that the equations for the dilaton in (7.16), the metric factor  $\rho$  in (7.18), and the metric factors  $f_1, f_2$ , and  $f_4$  in (9.1) all give regular results. The requirements on  $W, N_1$  and  $N_2$  may be condensed to the single additional requirement

**(R5)**  $(-WN_1) > 0$  throughout  $\Sigma$ .

This follows from the fact  $W$  or  $N_1$  may vanish if and only if their product vanishes. Secondly,  $N_2$  may vanish only if  $W$  vanishes.

### 9.3 Dirichlet or Neumann conditions and regularity on $\partial\Sigma$

Remarkably, the non-linear inequalities of (9.3) admit a linearization, similar to the one derived in [11]. There, the boundary condition  $W = 0$  was decomposed into a sequence of alternating Neumann and Dirichlet boundary conditions for the harmonic functions  $h_1$  and  $h_2$ . An analogous mechanism will now be identified for the problem at hand.

To this end, we assume that  $h_1$  admits a Taylor expansion in powers of  $y$  away from the boundary at  $y = 0$ . We have the following expansion,

$$h_1 = a_0(x) + a_1(x)y + a_2(x)y^2 + a_3(x)y^3 + \mathcal{O}(y^4) \tag{9.4}$$

where  $a_0, a_1, a_2, a_3$  are real functions of  $x$ . Since  $h_1$  is harmonic, it satisfies  $(\partial_x^2 + \partial_y^2)h_1 = 0$ . It follows that  $a_0(x)$  completely determines the coefficients of the terms with even powers of  $y$ , while  $a_1(x)$  completely determines the coefficients of the terms with odd powers of  $y$ ; we have  $2a_2(x) = -\partial_x^2 a_0(x)$  and  $6a_3(x) = -\partial_x^2 a_1(x)$ .

Since  $N_2$  automatically vanishes at  $y = 0$ , regularity of the dilaton at  $y = 0$  requires that also  $N_1$  vanishes at  $y = 0$ . This requirement may be enforced on the Taylor expansion of  $N_1$ ,

$$N_1 = -\frac{1}{4}a_0^2 a_1 - \frac{1}{4}\left(a_0 a_1^2 + 2a_0^2 a_2 - a_0(\partial_x a_0)^2\right)y + \mathcal{O}(y^2) \tag{9.5}$$

We see that we must have either  $a_0 = 0$  or  $a_1 = 0$ . Equivalently, at  $y = 0$ , the harmonic function  $h_1$  satisfies either  $\partial_y h_1 = 0$ , or  $h_1 = 0$ , i.e. it satisfies either Neumann or *vanishing* Dirichlet boundary conditions on  $\partial\Sigma$ . This result is analogous to the boundary value problem derived in the  $AdS_4$  case in [11].

We now verify both sets of boundary conditions yield regular solutions on  $\partial\Sigma$ . This may be verified directly from the Taylor expansion. In the case of Neumann boundary conditions,  $W, N_1$  and  $N_2$  have the following expansions,

$$W = 2a_2 y + \mathcal{O}(y^3)$$

$$\begin{aligned}
 N_1 &= -\frac{1}{2}a_0^2 a_2 y + \frac{1}{4}a_0(\partial_x a_0)^2 y + \mathcal{O}(y^3) \\
 N_2 &= \frac{1}{16}a_0 y + \mathcal{O}(y^3)
 \end{aligned}
 \tag{9.6}$$

In the case of *vanishing* Dirichlet boundary conditions, with  $a_0 = 0$ ,  $N_1$  and  $N_2$  have the following expansions,

$$\begin{aligned}
 W &= a_1 + \mathcal{O}(y^2) \\
 N_1 &= \frac{1}{2}a_1^2 a_3 y^4 + \frac{1}{4}a_1(\partial_x a_1)^2 y^4 + \mathcal{O}(y^6) \\
 N_2 &= -\frac{1}{8}a_3 y^4 + \mathcal{O}(y^6)
 \end{aligned}
 \tag{9.7}$$

Using either set of expressions, it can be verified that the equations for the dilaton in (7.16), the metric factor  $\rho$  in (7.18), and the metric factors  $f_1$ ,  $f_2$ , and  $f_4$  in (7.26) all give regular results in the limit  $y \rightarrow 0$ .

## 10. Hyperelliptic solutions

In this section, we shall present the general hyperelliptic form of regular solutions to the BPS equations. From the point of view of the Riemann surface  $\Sigma$ , the  $AdS_5 \times S^5$  solution corresponded to  $\Sigma$  being a (half-) plane, and the harmonic functions  $h_1$  and  $h_2$  obeying alternating Neumann and vanishing Dirichlet boundary conditions. The construction to be given in this section will generalize  $\Sigma$  to a hyperelliptic Riemann surface of general genus with boundary, where the harmonic functions obey generalized alternating Neumann and vanishing Dirichlet boundary conditions.

The starting point of the construction will be a hyperelliptic Riemann surface of arbitrary genus  $g$ , with real branch points  $e_1, e_2, \dots, e_{2g+1}, e_{2g+2} = \infty$ , and the following associated hyperelliptic curve,

$$s^2 = (u - e_1) \prod_{i=1}^g (u - e_{2i})(u - e_{2i+1})
 \tag{10.1}$$

For later convenience, we choose the branch points to be ordered as follow,

$$e_{2g+1} < e_{2g} < \dots < e_2 < e_1
 \tag{10.2}$$

Physical considerations underly the choice of possible differentials  $\partial h_1$  and  $\partial h_2$ .

First and foremost, the gravity dual to a Wilson loop should have only a single asymptotic  $AdS_5 \times S^5$  region. In the formulation (8.15) of  $AdS_5 \times S^5$ , this corresponds to the two double poles at  $(u_0, \pm s(u_0))$ . Thus, we shall require that the differentials  $\partial h_1$  and  $\partial h_2$  in the hyperelliptic generalization have precisely the same double pole structure as  $AdS_5 \times S^5$  had. We shall without loss of generality keep the ordering  $e_1 < u_0$ .

Second, the harmonic function  $h_2$  may be used for a description of the boundary of  $\Sigma$ . As argued in the previous section, throughout the boundary, we have  $h_2 = 0$ , and this

equation in turn describes the boundary completely. Thus, we shall leave  $h_2$  unmodified from its  $AdS_5 \times S^5$  expression in (8.15).

Third, the differential  $\partial h_1$  should be generalized in such a way that it remain regular, except at the double poles  $u_0$ . Assembling all of the above requirements, we arrive at the following differentials,

$$\begin{aligned}\partial h_1 &= -i \frac{P(u) du}{(u - u_0)^2 s(u)} \\ \partial h_2 &= i \frac{du}{(u - u_0)^2}\end{aligned}\tag{10.3}$$

Here,  $P(u)$  is a polynomial in  $u$ . Since  $h_1$  will obey alternating Neumann and vanishing Dirichlet boundary conditions on  $\partial\Sigma$ , represented here by the real line  $\text{Im}(u) = 0$ , the polynomial  $P$  must have either only imaginary or real coefficients. Without loss of generality, we may choose the coefficients to be all real. Since  $\partial h_1$  should be regular at the branch point at  $\infty$ , the polynomial  $P$  must have degree at most  $g + 1$ . In analogy with the  $AdS_5 \times S^5$  solution,  $\partial h_1$  should be non-vanishing at  $\infty$ , so that  $P(u)$  must be exactly of degree  $g + 1$ .

### 10.1 Counting parameters

At this point, it is useful to count the number of parameters of the hyperelliptic Ansatz. By the regularity arguments exhibited in 9.1, the function  $h_1$  is required to vanish on the real axis in the regions with Dirichlet boundary conditions. There are exactly  $g + 1$  such regions which gives  $g$  period relations. The polynomial  $P$  has  $g + 1$  free parameters, while there are  $2g - 1$  parameters  $e_i$  (up to overall  $SL(2, \mathbf{R})$  conformal rotations of  $\Sigma$ ) and the parameter  $u_0$ , yielding a total of  $3g + 1$  parameters. After solving the  $g + 1$  constraints, this leaves  $2g$  free parameters. In addition, there are 3 parameters for  $SU(1, 1)$  rotations to the general solution with axion, 1 for the overall dilaton shift, and 1 for the overall radius. The total is  $2g + 5$ . When  $g = 0$ , the dilaton and axion are constant so there are a total of only 3 parameters. This counting agrees with the result of subsection 8.2 that  $AdS_5 \times S^5$  is the only non-singular solution with  $g = 0$ .

### 10.2 Regularity conditions, $W > 0$

In this subsection, we shall enforce two of the regularity conditions, arrived at in section 9, namely  $W > 0$  and  $h_1, h_2 > 0$ . From (10.3), we readily evaluate  $W$  to be,

$$W = -\frac{1}{|u - u_0|^4} \left( \frac{P(u)}{s(u)} + \text{c.c.} \right)\tag{10.4}$$

Examining the behavior of  $W$  in the neighborhood of a putative complex zero  $u_i$  of  $P(u)$ , with  $\text{Im}(u_i) < 0$ , it is clear that  $W$  cannot maintain a constant sign circling around  $u_i$ . (This argument was made fully explicit in [11].) Thus, all  $g + 1$  zeros of  $P(u)$  must be real, and we shall denote them by  $\alpha_b$  for  $b = 1, \dots, g + 1$ , so that

$$P(u) = \prod_{b=1}^{g+1} (u - \alpha_b)\tag{10.5}$$

Reality of the zeros  $\alpha_b$  does not by itself guarantee positive  $W$ . To enforce positivity of  $W$  on the real axis, we need the behavior of  $s(u)$  on the the real axis. Clearly, the phase of  $s(u)$  changes by a factor of  $i$  upon traversing any branch point. This behavior is given by

$$\begin{aligned}
 s(u)/|s(u)| = -1 & & u \in \mathcal{U}_1 \equiv ]e_1, +\infty[ \cup \bigcup_{j=1}^{n_1} ]e_{4j+1}, e_{4j}] \\
 s(u)/|s(u)| = +i & & u \in \mathcal{U}_2 \equiv \bigcup_{j=0}^{n_2} ]e_{4j+2}, e_{4j+1}[ \\
 s(u)/|s(u)| = +1 & & u \in \mathcal{U}_3 \equiv \bigcup_{j=0}^{n_3} ]e_{4j+3}, e_{4j+2}] \\
 s(u)/|s(u)| = -i & & u \in \mathcal{U}_4 \equiv \bigcup_{j=0}^{n_4} ]e_{4j+4}, e_{4j+3}[
 \end{aligned} \tag{10.6}$$

The upper limits of these unions are given by

$$\begin{aligned}
 n_1 &= [g/2] \\
 n_2 &= [(2g - 1)/4] + n_{-\infty} \\
 n_3 &= [(g - 1)/2] \\
 n_4 &= [(2g - 3)/2] + 1 - n_{-\infty}
 \end{aligned} \tag{10.7}$$

where  $n_{-\infty} = 1$  when  $g$  is even and  $n_{-\infty} = 0$  when  $g$  is odd, and  $e_{2g+2} = -\infty$ . The analysis of  $W > 0$  on the real axis proceeds as follows.

When  $u \in \mathcal{U}_2$  or  $u \in \mathcal{U}_4$ ,  $s(u)$  is imaginary while  $P(u)$  is real, so that we have  $W = 0$  automatically on those segments. Using the analysis of section 9.2, this is the boundary component on which the sphere  $S^2$  shrinks to zero;  $W > 0$  places no restrictions here.

When  $u \in \mathcal{U}_1$ , or  $u \in \mathcal{U}_3$  both  $s(u)$  and  $P(u)$  are real, and  $W > 0$  imposes the conditions,

$$\begin{aligned}
 u \in \mathcal{U}_1 & & P(u) \geq 0 \\
 u \in \mathcal{U}_3 & & P(u) \leq 0
 \end{aligned} \tag{10.8}$$

This means that  $P(u)$  changes sign across each interval in  $\mathcal{U}_2$  and each interval in  $\mathcal{U}_4$ , so that it must have an odd number of zeros in each of these intervals. The total number of intervals in  $\mathcal{U}_2 \cup \mathcal{U}_4$  is precisely  $g + 1$ . Since  $P(u)$  has at most  $g + 1$  zeros, we find that  $P(u)$  has no zeros in  $\mathcal{U}_1 \cup \mathcal{U}_3$  and has precisely one zero in each interval in  $\mathcal{U}_2 \cup \mathcal{U}_4$ . Thus, we have a *unique ordering of the branch points and zeros of  $P(u)$* , given by,

$$\alpha_{g+1} < e_{2g+1} < \cdots < e_{2b+1} < e_{2b} < \alpha_b < e_{2b-1} < e_{2b-2} < \cdots < e_2 < \alpha_1 < e_1 < u \tag{10.9}$$

for  $b = 2, \dots, g$ . This ordering is a necessary condition for  $W > 0$ . We have no general proof that the ordering conditions are also sufficient, though they will appear to be in the elliptic case, to be treated in detail in the subsequent section.

### 10.3 Regularity conditions, $h_1, h_2 > 0$

The function  $h_2$  is readily obtained by integrating the differential  $\partial h_2$  and enforcing the boundary condition  $h_2 = 0$  at  $\partial\Sigma$  and we find,

$$h_2(u) = -\frac{i}{u - u_0} + \frac{i}{\bar{u} - u_0} = \frac{i(u - \bar{u})}{|u - u_0|^2} \quad (10.10)$$

and this is manifestly positive in the lower half  $u$  plane. Positivity of  $h_1$  requires that  $\text{Im}(\partial h_1) < 0$  along the  $h_1$  Dirichlet intervals  $\mathcal{U}_1$  and  $\mathcal{U}_3$ . This condition precisely coincides with the condition  $W > 0$ , as is also clear from the form of  $W$  given in (9.3).

### 10.4 Vanishing Dirichlet conditions

By construction, the function  $h_2$  vanishes all along the boundary  $\partial\Sigma$ . The only vanishing Dirichlet condition which is not automatic is on  $h_1$ . It is enforced by the vanishing of the following line integrals that join consecutive Dirichlet segments for  $h_1$ ,

$$\int_{e_{2j}}^{e_{2j-1}} \partial h_1 = 0 \quad j = 1, \dots, g+1 \quad (10.11)$$

where  $e_{2g+2} = -\infty$ . These integrals are automatically real since by definition  $\partial h_1$  has an overall factor of  $i$  in (10.3), and on the integration intervals,  $s(u)$  is purely imaginary.

### 10.5 Homology 3-spheres and RR 3-form charges

The  $g$  homology 3-spheres of the genus  $g$  solutions are given by

$$S_b^3 = [e_{2b+1}, e_{2b}] \times_f S^2 \quad b = 1, \dots, g \quad (10.12)$$

and the associated RR 3-form charge is given by

$$C_b = \int_{S_b^3} C_{(3)} = 8\pi \int_{e_{2b+1}}^{e_{2b}} d\tilde{h}_1 = 8\pi i \int_{e_{2b+1}}^{e_{2b}} \partial h_1 + \text{c.c.} \quad (10.13)$$

This allows us to write down an explicit formula for these charges,

$$C_b = 16\pi \int_{e_{2b+1}}^{e_{2b}} \frac{P(u) du}{(u - u_0)^2 s(u)} \quad (10.14)$$

These integrands are manifestly real and of uniform sign, and thus the charges are real and non-vanishing.

## 11. Elliptic solutions

In this section, we work out the genus 1 case in complete detail, and show that regular solutions do indeed exist. We shall also evaluate their 3-form and 5-form charges and identify the corresponding fluxes.

The starting point for the genus 1 construction is the elliptic curve, and its uniformization by the Weierstrass  $\wp$ -function,

$$\begin{aligned} s(u)^2 &= (u - e_1)(u - e_2)(u - e_3) \\ \wp'(z)^2 &= 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3) \end{aligned} \tag{11.1}$$

Here, we have  $u = \wp(z)$  and the sign of  $s(u)$  is chosen so that  $2s(u) = \wp'(z)$ . Without loss of generality, we may translate  $u$  and the branch points  $e_i$  by an overall constant, and choose  $e_1 + e_2 + e_3 = 0$ . In terms of the half-periods  $\omega_1$  (real),  $\omega_2 = \omega_1 + \omega_3$ , and  $\omega_3$  (purely imaginary), we have  $e_i = \wp(\omega_i)$  for  $i = 1, 2, 3$ . We shall also make use of the Weierstrass  $\zeta(z)$ - and  $\sigma(z)$ -functions, which are defined, in terms of  $\wp(z)$  by

$$\begin{aligned} \zeta'(z) &= -\wp(z) \\ \zeta(z) &= \sigma'(z)/\sigma(z) \end{aligned} \tag{11.2}$$

and the requirement that  $\sigma(z) = z + \mathcal{O}(z^3)$ .

### 11.1 Elliptic differentials

For the genus 1 case,  $P(u)$  has two real zeros,  $\alpha_1$  and  $\alpha_2$ , subject to the ordering condition,

$$\alpha_2 < e_3 < e_2 < \alpha_1 < e_1 < u_0 \tag{11.3}$$

To completely integrate the elliptic case, it will turn out to be convenient to use the following alternative parametrization of  $P(u) = (u - \alpha_1)(u - \alpha_2)$ ,

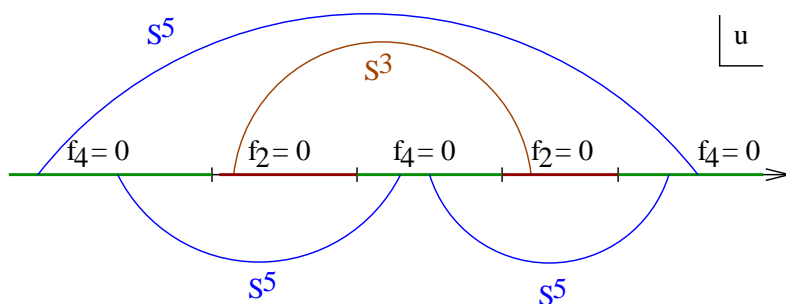
$$\begin{aligned} P(u) &= (u - u_0)^2 + B_1(u - u_0) + B_2 \\ B_1 &= 2u_0 - \alpha_1 - \alpha_2 \\ B_2 &= (u_0 - \alpha_1)(u_0 - \alpha_2) \end{aligned} \tag{11.4}$$

The constants  $B_1$  and  $B_2$  are real, and in view of the relative ordering of the branch points and zeros, we have  $B_1 > 0$  and  $B_2 > 0$ . The differentials  $\partial h_1$  and  $\partial h_2$  are then uniquely specified by  $\alpha_1, \alpha_2$  and  $u_0$ , and given by,

$$\begin{aligned} \partial h_1 &= -i \frac{(u - u_0)^2 + B_1(u - u_0) + B_2}{(u - u_0)^2} \frac{du}{s(u)} \\ \partial h_2 &= i \frac{du}{(u - u_0)^2} \end{aligned} \tag{11.5}$$

Given that  $u_0$  corresponds to the  $AdS_5 \times S^5$  limiting region,  $u_0$  must be real and lie in the interval  $[e_1, +\infty]$ . We define the point  $w_0$  by  $u_0 = \wp(w_0)$  and use the fact that  $u_0 \in [e_1, +\infty]$  to further locate  $w_0 \in [0, \omega_1]$ . The differentials may then be expressed in terms of the uniformization coordinates  $z$ ,

$$\begin{aligned} \partial h_1 &= -2i \left( \frac{B_2}{(\wp(z) - \wp(w_0))^2} + \frac{B_1}{\wp(z) - \wp(w_0)} + 1 \right) dz \\ \partial h_2 &= +i \frac{\wp'(z) dz}{(\wp(z) - \wp(w_0))^2} \end{aligned} \tag{11.6}$$



**Figure 3:** The geometry of the genus 1 solution.

## 11.2 Evaluation of the harmonic functions $h_1$ and $h_2$

It is straightforward to calculate  $h_2$ , and we find,

$$h_2 = -2i \left( \frac{1}{\wp(z) - \wp(w_0)} - \frac{1}{\overline{\wp(z)} - \wp(w_0)} \right) = \frac{2i(\wp(z) - \wp(\bar{z}))}{|\wp(z) - \wp(w_0)|^2} \quad (11.7)$$

To calculate  $h_1$ , we need the following integrals for  $n = 1, 2$ ,

$$I_n(z, w_0) \equiv \int \frac{dz}{(\wp(z) - \wp(w_0))^n} \quad (11.8)$$

These integrals may be calculated explicitly, using the addition formula for the  $\zeta$ -function,

$$\zeta(z + w_0) = \zeta(z) + \zeta(w_0) + \frac{1}{2} \frac{\wp'(z) - \wp'(w_0)}{\wp(z) - \wp(w_0)} \quad (11.9)$$

recast in the following form obtained by antisymmetrizing  $w_0$ ,

$$\zeta(z + w_0) - \zeta(z - w_0) = 2\zeta(w_0) - \frac{\wp'(w_0)}{\wp(z) - \wp(w_0)} \quad (11.10)$$

Integrating the right hand side in  $z$  yields the integral we need which is proportional to  $B_1$ ,

$$I_1(z, w_0) = \frac{1}{\wp'(w_0)} \left( \ln \sigma(z - w_0) - \ln \sigma(z + w_0) + 2\zeta(w_0)z \right) \quad (11.11)$$

up to an additive integration constant. The integral we need proportional to  $B_2$  may be derived from (11.11) by differentiating in  $w_0$ ,

$$I_2(z, w_0) = \frac{1}{\wp'(w_0)^2} \left( -\zeta(z - w_0) - \zeta(z + w_0) - 2z\wp(w_0) - \frac{1}{2} (12\wp(w_0)^2 - g_2) I_1(z, w_0) \right) \quad (11.12)$$

where we have used the standard notation for the modular form defined by  $g_2 = 4(e_1e_2 + e_2e_3 + e_3e_1)$ . Putting all together, we obtain the following expression for  $h_1$ ,

$$h_1 = B_0 - 2iB_2 \left( I_2(z, w_0) - I_2(\bar{z}, w_0) \right) - 2iB_1 \left( I_1(z, w_0) - I_1(\bar{z}, w_0) \right) - 2i \left( z - \bar{z} \right) \quad (11.13)$$

where  $B_0$  is a real integration constant.



### 11.3 Vanishing Dirichlet boundary conditions

In order to have a non-singular solution, the harmonic functions must vanish on the Dirichlet parts of the boundary. Attention only needs to be paid to  $h_1$  since  $h_2$  always satisfies Dirichlet boundary conditions on the real  $z$ -axis. The boundary structure for  $h_1$  is

$$\begin{aligned}
 -\infty < u < e_3 &: & \partial h_1 = \text{real} & \quad \text{Neumann} \\
 e_3 < u < e_2 &: & \partial h_1 = \text{imag} & \quad \text{Dirichlet} \\
 e_2 < u < e_1 &: & \partial h_1 = \text{real} & \quad \text{Neumann} \\
 e_1 < u < \infty &: & \partial h_1 = \text{imag} & \quad \text{Dirichlet}
 \end{aligned} \tag{11.14}$$

On the  $z$  half-plane, the vanishing Dirichlet requirements are given by

$$h_1(\rho) = h_1(\rho + \omega_3) = 0 \quad \rho \in \mathbf{R} \tag{11.15}$$

Evaluating the harmonic function on the real axis gives

$$\begin{aligned}
 h_1 = B_0 - \frac{2i}{\wp'(w_0)} &\left( B_1 - \frac{B_2(12\wp(w_0)^2 - g_2)}{2\wp'(w_0)^2} \right) \\
 &\times \left( \ln \sigma(z - w_0) - \ln \sigma(\bar{z} - w_0) - \ln \sigma(z + w_0) + \ln \sigma(\bar{z} + w_0) \right)
 \end{aligned} \tag{11.16}$$

We leave in the log functions since discontinuities may arise from crossing branch cuts, all other functions are singly valued (when  $z$  is restricted to region (I) in the fundamental domain). To analyze the discontinuities, we first choose the branch cut for the log function to run along the negative real axis. Consider first the case when  $\text{Re}(z) > w$ . As  $z$  approaches the real axis  $\sigma(z \pm w_0)$  and  $\sigma(\bar{z} \pm w_0)$  have real parts greater than zero and imaginary parts above or below the real axis. Since the branch cut runs along the negative axis the log terms simply cancel each other. Requiring  $h_1$  to vanish leads to the condition  $B_0 = 0$ . Now consider the case when  $\text{Re}(z) < w_0$ , for simplicity take  $\text{Re}(z) = 0$ . Now  $\sigma(z + w_0)$  and  $\sigma(\bar{z} + w_0)$  again have real parts on the positive real axis, but  $\sigma(z - w_0)$  and  $\sigma(\bar{z} - w_0)$  have real parts along the negative real axis. This is understood from the fact  $\sigma(z)$  is an odd function of  $z$ . Now as  $z$  approaches the real axis, the log terms involving  $z - w_0$  and  $\bar{z} - w_0$  pick up a discontinuity from the branch cut and no longer cancel. This is because the  $\sigma(z - w_0)$  approaches the real axis from above while  $\sigma(\bar{z} - w_0)$  approaches the real axis from below. This implies that their pre-factor must vanish which yields the condition

$$B_1 = \frac{B_2(12\wp(w_0)^2 - g_2)}{2\wp'(w_0)^2} \tag{11.17}$$

The above conditions are sufficient to ensure that  $h_1$  vanishes along the real axis. To enforce the vanishing of  $h_1$  on the second Dirichlet boundary it suffices to require  $h_1$  to vanish at  $\omega_3$ . This gives  $B_2$  as

$$1 = \frac{B_2}{\omega_3 \wp'(w_0)^2} \left( 2\omega_3 \wp(w_0) + \zeta(\omega_3 - w_0) + \zeta(\omega_3 + w_0) \right) \tag{11.18}$$

This relation may be simplified using the formula  $\zeta(\omega_3 - w_0) + \zeta(\omega_3 + w_0) = 2\zeta(\omega_3)$ , which can be derived from (11.10) and the fact  $\wp'(\omega_i) = 0$ . The Dirichlet constraints have now completely fixed the polynomial coefficients. There remains the two half-periods  $\omega_i$  and the point  $u_0$ . We may fix  $\omega_1 = 1$  by a scaling transformation, leaving  $\omega_3$  and  $u_0$  as the remaining parameters. In summary, after satisfying vanishing Dirichlet boundary conditions, the harmonic functions are given by

$$\begin{aligned} h_1 &= 2i \left( \zeta(z - w_0) + \zeta(z + w_0) - 2 \frac{\zeta(\omega_3)}{\omega_3} z - \text{c.c.} \right) \\ h_2 &= + \frac{i}{\wp'(w_0)} \left( \zeta(z + w_0) - \zeta(z - w_0) - \text{c.c.} \right) \end{aligned} \tag{11.19}$$

where we have dropped an overall positive coefficient of  $[2\wp(w_0) + 2\zeta(\omega_3)/\omega_3]^{-1}$  in the expression for  $h_1$ . By transformations 1. and 2. of subsection 7.7, this results in a constant shift of the dilaton, and an overall constant re-scaling of the metric and fluxes. In the expression for  $h_2$  we have made use of the identity (11.10) to rewrite the inverse of the Weierstrass function in terms of the Weierstrass zeta function.

#### 11.4 Positivity conditions and regularity

The harmonic function  $h_2$  is manifestly positive in view of the second relation in (11.7). In order to study the positivity of  $h_1$ , it will be useful to have a series expansion for the Weierstrass  $\zeta$ -function in terms of trigonometric functions,

$$\zeta(z) = z \frac{\zeta(\omega_3)}{\omega_3} + \frac{\pi}{2\omega_3} \sum_{m=-\infty}^{\infty} \cotg \left( \frac{\pi z}{2\omega_3} + \frac{m\pi\omega_1}{\omega_3} \right) \tag{11.20}$$

The harmonic function  $h_1$  is then given by,

$$h_1 = \frac{i\pi}{\omega_3} \sum_{m=-\infty}^{\infty} \left[ \cotg \left( \frac{\pi}{2\omega_3} (z + w_0 + 2m\omega_1) \right) + \cotg \left( \frac{\pi}{2\omega_3} (z - w_0 + 2m\omega_1) \right) \right] + \text{c.c.} \tag{11.21}$$

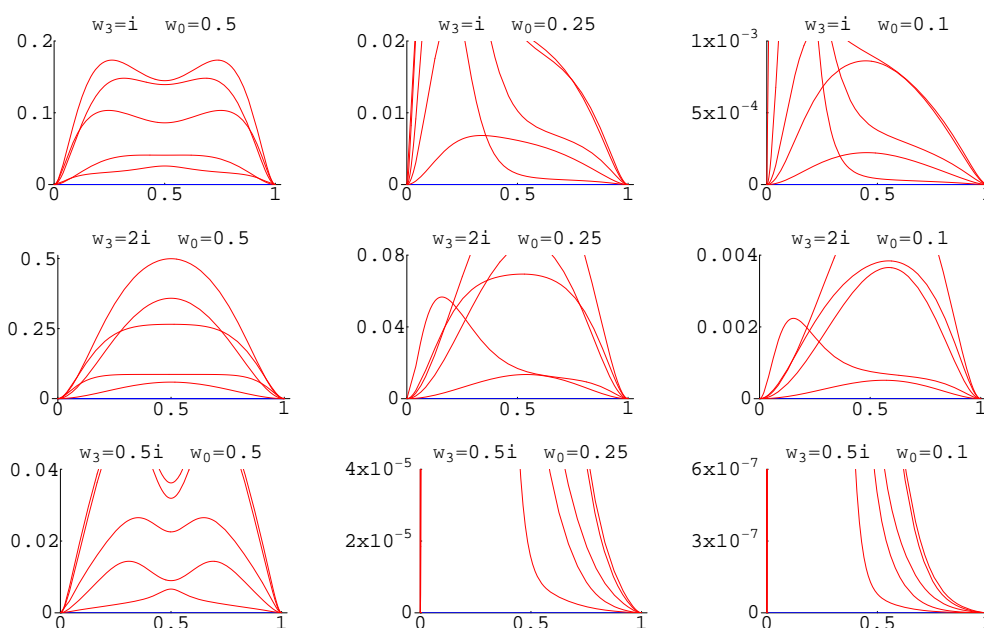
This may be simplified using the trigonometric identity,

$$\cotg u + \cotg \bar{u} = \frac{\sin(u + \bar{u})}{|\sin(u)|^2} \tag{11.22}$$

and we obtain,

$$\begin{aligned} h_1(z, \bar{z}) &= \frac{i\pi}{\omega_3} h(z, \bar{z}) \sin \left( \frac{\pi}{2\omega_3} (z - \bar{z}) \right) \\ h(z, \bar{z}) &= \sum_{m=-\infty}^{\infty} \left\{ \frac{1}{\left| \sin \left( \frac{\pi}{2\omega_3} (z + w_0 + 2m\omega_1) \right) \right|^2} + \frac{1}{\left| \sin \left( \frac{\pi}{2\omega_3} (z - w_0 + 2m\omega_1) \right) \right|^2} \right\} \end{aligned} \tag{11.23}$$

The harmonic function  $h_1$  is manifestly positive. We have now satisfied the regularity requirements (R1)-(R4) of section 9. It still remains to satisfy requirement (R5), which is



**Figure 4:** Plots of  $(-N_1W)$  versus  $\text{Re}(z)$  for values of  $\omega_3 = \frac{i}{2}, i, 2i$  and  $w_0 = \frac{1}{2}, \frac{1}{4}, \frac{1}{10}$ . The different lines in each plot are for evenly spaced values of  $\text{Im}(z)$  with  $0 < \text{Im}(z) < \text{Im}(\omega_3)$  (red) and  $\text{Im}(z) = 0, \text{Im}(\omega_3)$  (blue). On  $\partial\Sigma$ ,  $(-N_1W) = 0$  as expected.

$N_1W > 0$  throughout  $\Sigma$  and vanishes only on  $\partial\Sigma$ . While we have not been able to show this analytically, numerical investigations, shown <sup>4</sup> in figure 4, indicate that requirement (R5) holds in the elliptic case for any values of the parameters  $\omega_3, w_0$ .

### 11.4.1 Explicit solution with $\omega_3 = i$ and $w_0 = \frac{1}{2}$

Here, we present plots of the dilaton and the metric factors in the case  $\omega_3 = i$  and  $w_0 = \frac{1}{2}$ . It is clearly from figure 5 that the dilaton is not constant, so that the solution is different from  $AdS_5 \times S^5$ . The metric factor  $f_1$  never vanishes, while  $f_2$  and  $f_4$  vanish only on  $\partial\Sigma$ .

## 12. Collapse of branch cuts

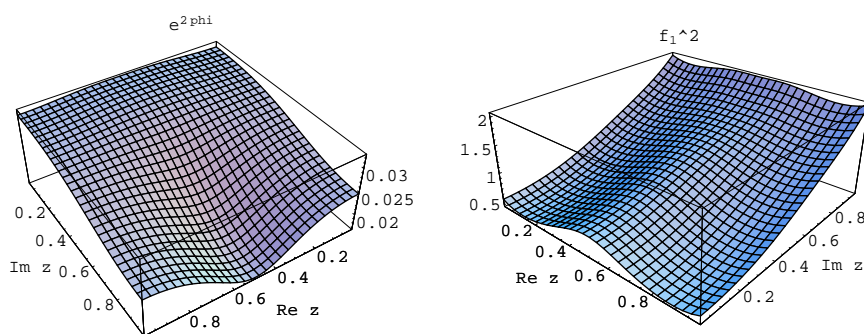
We recall the general ordering condition (10.9) on the branch points and zeros of the genus  $g$  hyperelliptic solution,

$$\alpha_{g+1} < e_{2g+1} < \dots < e_{2b+1} < e_{2b} < \alpha_b < \dots < e_1 < u_0 \tag{12.1}$$

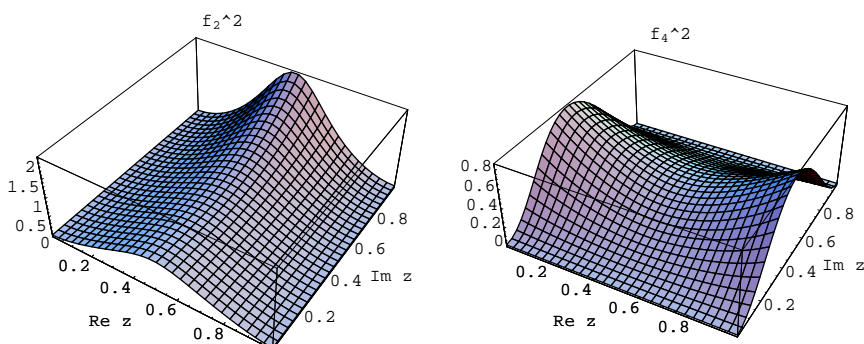
for  $b = 1, \dots, g$ . From this condition, it is clear that consecutive pairs of branch points come in two varieties, according to whether their fibration involves an  $S^2$  or an  $S^4$ ,

$$\begin{aligned} [e_{2b+1}, e_{2b}] \times_f S^2 & \quad \text{D5 - brane} \\ [e_{2b}, e_{2b-1}] \times_f S^4 & \quad \text{D3 - brane} \end{aligned} \tag{12.2}$$

<sup>4</sup>In figure 4, a multiplicative factor of  $|z - w_0|^{10}$  has been included to regularize the singularity at  $z = w_0$  which corresponds to the asymptotic  $AdS_5 \times S^5$  region, while in figures 5 and 6, an analogous multiplicative factor of  $|z - w_0|^2$  has been included.



**Figure 5:** Plots of the dilaton (left) and the metric factor  $f_1^2$  (right) for the elliptic case with  $\omega_3 = i$  and  $w_0 = \frac{1}{2}$ .



**Figure 6:** Plots of the metric factor  $f_2^2$  (left) and the metric factor  $f_4^2$  (right) for the elliptic case with  $\omega_3 = i$  and  $w_0 = \frac{1}{2}$ .

Here, the product  $\times_f$  stands for a fibration, not for a product of sets. The collapse of a branch point in each variety corresponds to the collapse of the corresponding  $S^3$  or  $S^5$ , since the radii of  $S^2$  and  $S^4$  are always 1.

This situation is to be contrasted with the  $AdS_4 \times S^2 \times S^2 \times \Sigma$  case in [11], where each segment between consecutive branch points always corresponded to a homology 3-sphere. The shrinking of one of these  $S^3$  in a genus  $g$  solution was regular or singular depending on how the limit of the real zeros was taken. We showed in [11] that a singular limit exists in which naked D5-branes and NS5-branes are recovered. These naked solutions, in turn admitted a probe limit, which reflected the familiar singularity structure of the D5- and NS5-branes in flat space-time.

The situation for the  $AdS_2 \times S^2 \times S^4 \times \Sigma$  solutions of this paper is rather different. We shall show that the limit of collapsing branch cuts of a regular genus  $g$  hyperelliptic solution only leads to the non-singular solution of genus  $g - 1$ , but there is no room for naked D5-branes, NS5-branes, or D3-branes. To keep calculations as explicit as possible, we illustrate this effect on the genus 1 solution.

### 12.1 The genus 1 solution

We study the collapse of a branch cut in the genus 1 case, where the ordering of branch points and zeros is given as follows,

$$\alpha_2 < e_3 < e_2 < \alpha_1 < e_1 < u_0 \tag{12.3}$$

The collapse of the segment  $[e_3, e_2]$  corresponds to a collapsing  $S^3$ , while the collapse of the segment  $[e_2, e_1]$  (or of  $[-\infty, e_3]$ ) corresponds to a collapsing  $S^5$ . The behaviors of these two collapses are qualitatively different. We recall the expression for the differential  $\partial h_1$  and the harmonic function  $h_2$  in the  $u$ -coordinates,

$$\begin{aligned} \partial h_1 &= -i \frac{(u - \alpha_1)(u - \alpha_2)du}{(u - u_0)^2 \sqrt{(u - e_1)(u - e_2)(u - e_3)}} \\ h_2 &= -i \left( \frac{1}{u - u_0} - \frac{1}{u - e_1} \right) \end{aligned} \tag{12.4}$$

where the square root is taken to be negative for  $u > e_1$ .

### 12.2 The collapse of $[e_2, e_1]$

As  $e_1 - e_2 \rightarrow 0$ , we necessarily also have  $\alpha_1 \rightarrow e_1 = e_2$ . As a result, the differential  $\partial h_1$  simplifies considerably,

$$\partial h_1 = -i \frac{(u - \alpha_2)du}{(u - u_0)^2 \sqrt{u - e_3}} \tag{12.5}$$

Comparison with (8.15) immediately reveals that this is just the  $AdS_5 \times S^5$  solution at genus 0, with  $\alpha_2 = e_3 - 1/2$ . In figure 1, this collapse corresponds to the upper  $S^5$  shrinking to 0. Note that in the process, the homology 3-sphere disappears as well.

### 12.3 The collapse of $[e_3, e_2]$

By an overall translation of  $u$ , we may fix  $e_1 = 0$ , and it will be convenient to designate the shifted branch points by  $k^2 \equiv -e_3 = -e_2$ , with  $k > 0$ . The collapse of the branch points for generic  $\alpha_1$  and  $\alpha_2$  would leave a pole located in a region with vanishing  $S^2$ . But  $\alpha_1$  and  $\alpha_2$  are actually completely determined in terms of  $u_0$  and the moduli. Here, we shall recover these relations directly from the genus 1 solution. To uniformize the square root, we introduce the coordinate  $w^2 \equiv u$ , with  $w_0^2 \equiv u_0$ , and  $w$  takes values in the second quadrant,  $\text{Re}(w) < 0$  and  $\text{Im}(w) > 0$ . In these coordinates, the differential becomes simply

$$\partial h_1 = -2i \frac{(w^2 - \alpha_1)(w^2 - \alpha_2)}{(w^2 - w_0^2)^2 (w^2 + k^2)} dw \tag{12.6}$$

To integrate we decompose in partial fractions

$$\partial h_1 = -2i \left( \frac{A}{(w - w_0)^2} + \frac{A}{(w + w_0)^2} + \frac{B}{w - w_0} - \frac{B}{w + w_0} + \frac{iC}{w + ik} - \frac{iC}{w - ik} \right) du \tag{12.7}$$

where the constant coefficients are given by

$$A = \frac{(w_0^2 - \alpha_1)(w_0^2 - \alpha_2)}{4w_0^2(w_0^2 + k^2)} > 0 \quad B = \frac{1}{2w_0}(1 - 2A - 2kC) \quad C = \frac{(k^2 + \alpha_1)(k^2 + \alpha_2)}{2k(w_0^2 + k^2)^2} < 0 \quad (12.8)$$

Integrating the harmonic function, we obtain

$$\begin{aligned} h_1 &= -2i \left[ D - A \left( \frac{1}{w - w_0} + \frac{1}{w + w_0} \right) + B \ln \left( \frac{w - w_0}{w + w_0} \right) - iC \ln \left( \frac{w - ik}{w + ik} \right) - \text{c.c.} \right] \\ h_2 &= -i \left[ \frac{1}{w^2 - w_0^2} - \text{c.c.} \right] \end{aligned} \quad (12.9)$$

where  $D$  is a purely imaginary integration constant.

The boundary structure is the same as in the elliptic case (11.14), but with the branch points  $e_2$  and  $e_3$  identified. Translating to the  $w$ -plane,  $h_1$  satisfies Neumann conditions along the imaginary  $w$ -axis, and vanishing Dirichlet conditions along the real  $w$ -axis. In these coordinates, real infinity and imaginary infinity correspond to the same branch point in the  $u$ -plane, so  $h_1$  should also be required to vanish at imaginary infinity. The explicit conditions are

$$h_1(\rho) = 0 \quad \rho \in \mathbf{R} \quad h_1(i\rho) = 0 \quad \rho \rightarrow \infty \quad (12.10)$$

Vanishing at  $w = \infty$  clearly requires  $D = 0$ . Vanishing on the left and right of  $w_0$  requires that the coefficient of the branch cut discontinuity of  $\ln(w - w_0)$  vanish as well, so that we must have  $B = 0$ . Once  $B = D = 0$ , it is manifest that  $h_1$  vanishes at imaginary infinity, and that all of the vanishing Dirichlet conditions (12.10) are satisfied.

The term proportional to  $A > 0$  in  $h_1$  is easily seen to be positive for  $w$  in the second quadrant. But the term proportional to  $C < 0$  is negative there. In particular, as  $w \rightarrow ik$ , this term would make  $h_1$  tend to  $-\infty$ , producing a behavior which is unacceptable for a physical solution. Thus, we must have  $C = 0$ , which in turn requires that  $\alpha_1 = -k^2$  or  $\alpha_2 = -k^2$ . The vanishing of  $B$  now requires that  $2A = 1$ , which determined  $\alpha_2 = -w_0^2$  or  $\alpha_1 = -w_0^2$  respectively. In either case do we recover the genus 0 solution which is  $AdS_5 \times S^5$ .

## Acknowledgments

This work was supported in part by National Science Foundation (NSF) Physics Division grant PHY-04-56200.

## A. Clifford algebra basis adapted to the Ansatz

We use the convention  $\eta = \text{diag}[- + \cdots +]$ , and choose a basis for the Clifford algebra which is well-adapted to the  $AdS_2 \times S^2 \times S^4 \times \Sigma$  Ansatz, with the frame labeled as in (3.2),

$$\begin{aligned} \Gamma^\mu &= \gamma^\mu \otimes I_2 \otimes I_4 \otimes I_2 & \mu &= 0, 1 \\ \Gamma^i &= \gamma_{(1)} \otimes \gamma^i \otimes I_4 \otimes I_2 & i &= 2, 3 \\ \Gamma^m &= \gamma_{(1)} \otimes \gamma_{(2)} \otimes \gamma^m \otimes I_2 & m &= 4, 5, 6, 7 \\ \Gamma^a &= \gamma_{(1)} \otimes \gamma_{(2)} \otimes \gamma_{(3)} \otimes \gamma^a & a &= 8, 9 \end{aligned} \quad (A.1)$$

where a convenient basis for the lower-dimensional Clifford algebras is as follows,

$$\begin{aligned}
 i\gamma^0 &= \sigma^2 & \gamma^4 &= \sigma^1 \otimes I_2 & \gamma^8 &= \sigma^1 \\
 \gamma^1 &= \sigma^1 & \gamma^5 &= \sigma^2 \otimes I_2 & \gamma^9 &= \sigma^2 \\
 \gamma^2 &= \sigma^2 & \gamma^6 &= \sigma^3 \otimes \sigma^1 \\
 \gamma^3 &= \sigma^1 & \gamma^7 &= \sigma^3 \otimes \sigma^2
 \end{aligned} \tag{A.2}$$

so that the chirality matrices take the form,

$$\begin{aligned}
 \gamma_{(1)} &= -\gamma^{01} = \sigma^3 & \gamma_{(3)} &= -\gamma^{4567} = \sigma^3 \otimes \sigma^3 \\
 \gamma_{(2)} &= i\gamma^{23} = \sigma^3 & \gamma_{(4)} &= -i\gamma^{89} = \sigma^3
 \end{aligned} \tag{A.3}$$

We shall also need the chirality matrices on the various components of  $AdS_2 \times S^2 \times S^4 \times \Sigma$ , and they are chosen as follows,

$$\begin{aligned}
 \Gamma_{(1)} &= -\Gamma^{01} = \gamma_{(1)} \otimes I_2 \otimes I_4 \otimes I_2 \\
 \Gamma_{(2)} &= +i\Gamma^{23} = I_2 \otimes \gamma_{(2)} \otimes I_4 \otimes I_2 \\
 \Gamma_{(3)} &= -\Gamma^{4567} = I_2 \otimes I_2 \otimes \gamma_{(3)} \otimes I_2 \\
 \Gamma_{(4)} &= -i\Gamma^{89} = I_2 \otimes I_2 \otimes I_4 \otimes \gamma_{(4)}
 \end{aligned} \tag{A.4}$$

The 10-dimensional chirality matrix in this basis is given by

$$\Gamma^{11} = \Gamma^{0123456789} = \Gamma_{(1)}\Gamma_{(2)}\Gamma_{(3)}\Gamma_{(4)} = \gamma_{(1)} \otimes \gamma_{(2)} \otimes \gamma_{(3)} \otimes \gamma_{(4)} \tag{A.5}$$

The complex conjugation matrices in each component are defined by

$$\begin{aligned}
 (\gamma^\mu)^* &= +B_{(1)}\gamma^\mu B_{(1)}^{-1} & (B_{(1)})^* B_{(1)} &= +I_2 & B_{(1)} &= I_2 \\
 (\gamma^i)^* &= -B_{(2)}\gamma^i B_{(2)}^{-1} & (B_{(2)})^* B_{(2)} &= -I_2 & B_{(2)} &= \gamma^2 = \sigma^2 \\
 (\gamma^m)^* &= -B_{(3)}\gamma^m B_{(3)}^{-1} & (B_{(3)})^* B_{(3)} &= -I_4 & B_{(3)} &= i\gamma^{46} = \sigma^2 \otimes \sigma^1 \\
 (\gamma^a)^* &= -B_{(4)}\gamma^a B_{(4)}^{-1} & (B_{(4)})^* B_{(4)} &= -I_2 & B_{(4)} &= \gamma^9 = \sigma^2
 \end{aligned} \tag{A.6}$$

where in the last column we have also listed the form of these matrices in our particular basis. It is also useful to note how the  $B_{(i)}$  commute past the chirality matrices  $\gamma_{(i)}$

$$\begin{aligned}
 B_{(1)}\gamma_{(1)} &= +\gamma_{(1)}B_{(1)} \\
 B_{(2)}\gamma_{(2)} &= -\gamma_{(2)}B_{(2)} \\
 B_{(3)}\gamma_{(3)} &= +\gamma_{(3)}B_{(3)} \\
 B_{(4)}\gamma_{(4)} &= -\gamma_{(4)}B_{(4)}
 \end{aligned} \tag{A.7}$$

The 10-dimensional complex conjugation matrix  $\mathcal{B}$  is defined by  $(\Gamma^M)^* = \mathcal{B}\Gamma^M\mathcal{B}^{-1}$  and  $\mathcal{B}\mathcal{B}^* = I$ , and in this basis is given by

$$\begin{aligned}
 \mathcal{B} &= -\Gamma^{2579} = +iB_{(1)} \otimes \gamma_{(2)}B_{(2)} \otimes B_{(3)} \otimes B_{(4)} \\
 &= I_2 \otimes \sigma^1 \otimes \sigma^2 \otimes \sigma^1 \otimes \sigma^2
 \end{aligned} \tag{A.8}$$

and satisfies  $\mathcal{B}^* = \mathcal{B}$ , and  $\mathcal{B}^2 = I$ .

It is useful to check that the self-duality of  $F_{(5)}$  is compatible with the chirality conventions of the  $\Gamma$ -matrices. To this end, we evaluate

$$\begin{aligned} \Gamma^{0123} &= i\Gamma_{(1)}\Gamma_{(2)} & \frac{1}{5!}(F_{(5)} \cdot \Gamma) &= -if_a\Gamma_{(1)}\Gamma_{(2)}\Gamma^a (I + \Gamma^{11}) \\ \Gamma^{4567} &= -\Gamma_{(3)} \end{aligned} \tag{A.9}$$

which has the correct projection properties. Here, we have used,  $\varepsilon^{ab}\gamma^b = i\gamma^a\gamma_{(4)}$ ,  $a, b = 8, 9$ .

## B. The geometry of Killing spinors

We review the relation between the Killing spinor equation and the parallel transport equation in the presence of a flat connection with torsion on  $AdS_2$ , and even-dimensional spheres.

### B.1 Minkowski $AdS_2$

For Minkowski signature, we have  $AdS_2 = SO(2,1)/SO(1,1)$ . The Clifford algebra of  $SO(2,1)$  is built from the Clifford generators  $\gamma^\mu$ , with  $\mu = 0, 1$ ,

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \quad \eta = \text{diag}[-+ ] \tag{B.1}$$

and the matrix,  $\gamma^\sharp \equiv i\gamma^0\gamma^1$ , which is proportional to the chirality matrix, but has  $(\gamma^\sharp)^2 = -1$ . The matrices  $\gamma^\sharp, \gamma^0, \gamma^1$  form a 2-dimensional representation of the Clifford algebra,

$$\{\gamma^{\bar{\mu}}, \gamma^{\bar{\nu}}\} = 2\bar{\eta}^{\bar{\mu}\bar{\nu}} \quad \bar{\eta} = \text{diag}[- - + ] \tag{B.2}$$

for  $\bar{\mu}, \bar{\nu} = \sharp, 0, 1$ . The corresponding Maurer-Cartan form on  $SO(2,1)$  is given by

$$\omega^{(t)} = V^{-1}dV = \frac{1}{4}\omega_{\bar{\mu}\bar{\nu}}^{(t)}\gamma^{\bar{\mu}\bar{\nu}} \quad V \in SO(2,1) \tag{B.3}$$

It obviously satisfies the Maurer-Cartan equations,  $d\omega^{(t)} + \omega^{(t)} \wedge \omega^{(t)} = 0$ . We decompose  $\omega^{(t)}$  onto the  $SO(1,1)$  connection and cotangent space frame on  $AdS_2$

$$\omega^{(t)} = \frac{1}{4}\omega_{\mu\nu}\gamma^{\mu\nu} + \frac{1}{2}e_\mu\gamma^\mu\gamma^\sharp \quad \begin{cases} \omega_{\mu\nu} \equiv \omega_{\mu\nu}^{(t)} & \mu, \nu = 0, 1 \\ e_\mu \equiv \omega_{\mu\sharp}^{(t)} & \mu = 0, 1 \end{cases} \tag{B.4}$$

The Maurer-Cartan equations  $d\omega^{(t)} + \omega^{(t)} \wedge \omega^{(t)} = 0$  for  $\omega^{(t)}$  imply the absence of torsion and the constancy of curvature. The Killing spinor equation coincides with the equation for parallel transport,

$$(d + V^{-1}dV) \varepsilon \left( d + \frac{1}{4}\omega_{\mu\nu}\gamma^{\mu\nu} + \frac{1}{2}\eta e_\mu\gamma^\mu\gamma^\sharp \right) \varepsilon = 0 \tag{B.5}$$

For  $\eta = +1$ , the general solution is given by  $\varepsilon_+ = V^{-1}\varepsilon_0$  and  $\varepsilon_0$  is constant, while for  $\eta = -1$ , the solution is  $\varepsilon_- = \gamma^\sharp\varepsilon_+$ .



## B.2 Even-dimensional spheres

The sphere is a coset,  $S^N = \text{SO}(N+1)/\text{SO}(N)$ , and we consider the case  $N$  even. The Clifford algebra of  $\text{SO}(N+1)$  is built from the Clifford generators  $\gamma^m$  of  $\text{SO}(N)$ ,

$$\{\gamma^m, \gamma^n\} = 2\delta^{mn} \quad (\text{B.6})$$

for  $m, n = 1, 2, \dots, N$ , supplemented with the chirality matrix,  $\gamma^\sharp \equiv (i)\gamma^{1\dots N}$ , where the  $(i)$  is chosen so that  $(\gamma^\sharp)^2 = +1$ ,

$$\{\gamma^{\bar{m}}, \gamma^{\bar{n}}\} = 2\delta^{\bar{m}\bar{n}} \quad (\text{B.7})$$

for  $\bar{m}, \bar{n} = \sharp, 1, 2, \dots, N$ . The corresponding Maurer-Cartan form on  $\text{SO}(N+1)$  is given by

$$\omega^{(t)} = V^{-1}dV = \frac{1}{4}\omega_{\bar{m}\bar{n}}^{(t)}\gamma^{\bar{m}\bar{n}} \quad V \in \text{SO}(N+1) \quad (\text{B.8})$$

It obviously satisfies the Maurer-Cartan equations,  $d\omega^{(t)} + \omega^{(t)} \wedge \omega^{(t)} = 0$ . We decompose  $\omega^{(t)}$  onto the  $\text{SO}(N)$  and  $S^N$  directions of cotangent space,

$$\omega^{(t)} = \frac{1}{4}\omega_{mn}\gamma^{mn} + \frac{1}{2}e_m\gamma^m\gamma^\sharp \quad \begin{cases} \omega_{mn} \equiv \omega_{mn}^{(t)} & m, n = 1, 2, \dots, N \\ e_m \equiv \omega_{m\sharp}^{(t)} & m = 1, 2, \dots, N \end{cases} \quad (\text{B.9})$$

The Maurer-Cartan equations  $d\omega^{(t)} + \omega^{(t)} \wedge \omega^{(t)} = 0$  for  $\omega^{(t)}$  imply the absence of torsion and the constancy of curvature. The Killing spinor equation coincides with the equation for parallel transport,

$$(d + V^{-1}dV)\varepsilon = \left(d + \frac{1}{4}\omega_{mn}\gamma^{mn} - \frac{1}{2}\eta e_m\gamma^m\gamma^\sharp\right)\varepsilon = 0 \quad (\text{B.10})$$

For  $\eta = -1$ , the general solution is given by  $\varepsilon_- = V^{-1}\varepsilon_0$  and  $\varepsilon_0$  is constant, while for  $\eta = +1$ , the solution is  $\varepsilon_+ = \gamma^\sharp\varepsilon_-$ .

## B.3 Explicit form of Killing equations on $AdS_2$ , $S^2$ , and $S^4$

- For  $AdS_2$  we have  $\gamma^0 = -i\sigma^2$ ,  $\gamma^1 = \sigma^1$ , and  $\gamma^\sharp = i\gamma_{(1)} = i\sigma^3$ . The Killing spinor equation is then, for  $\mu = 0, 1$ ,

$$\left(\hat{\nabla}_\mu - i\frac{\eta_1}{2}\gamma_\mu\gamma_{(1)}\right)\varepsilon = 0 \quad \Leftrightarrow \quad \left(\hat{\nabla}_\mu - \frac{\eta_1}{2}\gamma_\mu\right)\varepsilon' = 0 \quad (\text{B.11})$$

where  $\varepsilon' = e^{-i\frac{\pi}{4}\gamma_{(1)}}\varepsilon$ , and  $\eta_1 = \pm 1$ .

- For  $S_2$  we have  $\gamma^2 = \sigma^2$ ,  $\gamma^3 = \sigma^1$ , and  $\gamma^\sharp = \gamma_{(1)}$ . The Killing spinor equation is then, for  $i = 2, 3$ ,

$$\left(\hat{\nabla}_i - \frac{\eta_2}{2}\gamma_i\gamma_{(2)}\right)\varepsilon = 0 \quad \Leftrightarrow \quad \left(\hat{\nabla}_i - i\frac{\eta_2}{2}\gamma_i\right)\varepsilon' = 0 \quad (\text{B.12})$$

where  $\varepsilon' = e^{-i\frac{\pi}{4}\gamma_{(2)}}\varepsilon$ , and  $\eta_2 = \pm 1$ .

- For  $S^4$ , the  $\gamma^m$ ,  $m = 4, 5, 6, 7$  were given above and  $\gamma^\sharp = \gamma^{4567} = -\gamma_{(3)}$ . The Killing spinor equation is given by, for  $m = 4, 5, 6, 7$ ,

$$\left(\hat{\nabla}_m - \frac{\eta_3}{2}\gamma_m\gamma_{(3)}\right)\varepsilon = 0 \quad \Leftrightarrow \quad \left(\hat{\nabla}_m - i\frac{\eta_3}{2}\gamma_m\right)\varepsilon' = 0 \quad (\text{B.13})$$

where  $\varepsilon' = e^{-i\frac{\pi}{4}\gamma_{(3)}}\varepsilon$ , and  $\eta_3 = \pm 1$ .

### C. The reduced Bianchi identities and field equations

Using the differential form notation of section 1.2, it is straightforward to reduced the Bianchi identities to this Ansatz. We find,

$$\begin{aligned}
 0 &= dP - 2iQ \wedge P \\
 0 &= dQ + iP \wedge \bar{P} \\
 0 &= d\mathcal{G} + 2(d\ln f_1) \wedge \mathcal{G} - iQ \wedge \mathcal{G} + P \wedge \bar{\mathcal{G}} \\
 0 &= d\mathcal{H} + 2(d\ln f_2) \wedge \mathcal{H} - iQ \wedge \mathcal{H} - P \wedge \bar{\mathcal{H}} \\
 0 &= d(*_2\mathcal{F}) + 4(d\ln f_4) \wedge (*_2\mathcal{F}) \\
 0 &= d\mathcal{F} + 2(d\ln(f_1f_2)) \wedge \mathcal{F} + \frac{1}{8}(\mathcal{G} \wedge \bar{\mathcal{H}} + \bar{\mathcal{G}} \wedge \mathcal{H})
 \end{aligned} \tag{C.1}$$

The field equations of Type IIB supergravity, reduced to the two-parameter Ansatz of section 1.2 are given as follows. The BPS equations will imply that every solution may be mapped to one with vanishing axion and thus  $Q = 0$ , and  $g_a, h_a$  and  $p_a$  real.

Using the convention  $f^2 dB = d\phi$ , the dilaton equation becomes,

$$D^a D_a \phi + 2(D^a \phi) D_a \ln(f_1 f_2 f_4) - \frac{1}{4}(g_a g^a + h_a h^a) = 0 \tag{C.2}$$

The 3-form field equations reduce to the following two real equations,

$$\begin{aligned}
 D^a g_a + 2g_a D^a \ln(f_2 f_4^2) - p^a g_a + 4f_a h_a &= 0 \\
 D^a h_a + 2h_a D^a \ln(f_1 f_4^2) + p^a h_a + 4f_a g_a &= 0
 \end{aligned} \tag{C.3}$$

Finally, the Einstein equations, respectively for the components  $mn$ ,  $i_1 j_1$ ,  $i_2 j_2$ , and  $ab$  are as follows, (all other components must vanish by  $\text{SO}(2,3) \times \text{SO}(3) \times \text{SO}(3)$  symmetry),

$$\begin{aligned}
 0 &= +\frac{3}{f_4^2} - 3\frac{|D_a f_4|^2}{f_4^2} - 2\frac{D^a f_4 D_a (f_1 f_2)}{f_1 f_2 f_4} - \frac{D^a D_a f_4}{f_4} - 4f_a f^a - \frac{1}{8}g_a g^a + \frac{1}{8}h_a h^a \\
 0 &= -\frac{1}{f_1^2} - \frac{|D_a f_1|^2}{f_1^2} - 4\frac{D^a f_4 D_a f_1}{f_1 f_4} - 2\frac{D^a f_1 D_a f_2}{f_1 f_2} - \frac{D^a D_a f_1}{f_1} + 4f_a f^a + \frac{3}{8}g_a g^a + \frac{1}{8}h_a h^a \\
 0 &= \frac{1}{f_2^2} - \frac{|D_a f_2|^2}{f_2^2} - 4\frac{D^a f_4 D_a f_2}{f_2 f_4} - 2\frac{D^a f_1 D_a f_2}{f_1 f_2} - \frac{D^a D_a f_2}{f_2} + 4f_a f^a - \frac{1}{8}g_a g^a - \frac{3}{8}h_a h^a \\
 0 &= -4\frac{D_b D_a f_4}{f_4} - 2\frac{D_b D_a f_1}{f_1} - 2\frac{D_b D_a f_2}{f_2} + R^{(2)}\delta_{ab} - 2D_a \phi D_b \phi - 4\delta_{ab} f_c f^c + 8f_a f_b \\
 &\quad - \frac{1}{8}\delta_{ab}(g_c g^c - h_c h^c) - \frac{1}{2}(-g_a g_b + h_a h_b)
 \end{aligned} \tag{C.4}$$

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